THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA
BHASKARA CONTEST – FINAL - JUNIOR
Classes IX & X

Instructions:
1. Answer as many questions as possible.
2. Elegant and novel solutions will get extra credits.
3. Diagrams and explanations should be given wherever necessary.
4. Fill in FACE SLIP and your rough working should be in the answer book.
5. Maximum time allowed is THREE hours.
6. All questions carry equal marks.

Q.1 (a) Find all prime numbers p such that $4p^2 + 1$ and $6p^2 + 1$ are also primes.
(b) Determine real numbers $x$, $y$, $z$, $u$ such that
\[
\begin{align*}
xyz + xy + yz + zx + x + y + z &= 7 \\
yzu + yz + zu + uy + y + z + u &= 9 \\
zux + zu + ux + xz + z + u + x &= 9 \\
uxy + ux + xy + yu + u + x + y &= 9
\end{align*}
\]

Sol. (a) Let $p = 2$
\[
4p^2 + 1 = 4(2)^2 + 1 = 17
\]
\[
6p^2 + 1 = 6(2)^2 + 1 = 25 \text{ (not prime)}
\]
Let $p = 3$
\[
4(3)^2 + 1 = 37
\]
\[
6(3)^2 + 1 = 55 \text{ (not prime)}
\]
Let $p = 5$
\[
4(5)^2 + 1 = 101
\]
\[
6(5)^2 + 1 = 151
\]
So $4p^2 + 1$ and $6p^2 + 1$ both are prime for $p = 5$ we know that every square number is of the form $5m$, $5m + 1$ or $5m + 4$
Let take prime $p > 5$
So $p$ can be $5m + 1$ or $5m + 4$
Case (1)
$5m + 1$
\[
4p^2 + 1 = 4(5m + 1)^2 + 1 = 20k + 5 = 5(4k + 1) \quad \text{(A multiple of 5)}
\]
Case (2)
$5m + 4$
\[
6p^2 + 1 = 6(5m + 1)^2 + 1 = 30n + 25 = 5(6n + 5) \quad \text{(multiple of 5)}
\]
\[\therefore p = 5 \text{ is the only solution.}\]
(b) \[ xyz + xy + yz + zx + x + y + z = 7 \]
\[ xy(z + 1) + y(z + 1) + x(z + 1) + (z + 1) = 8 \]
\[ (z + 1)(xy + y + x + 1) = 8 \quad \text{...(1)} \]
Similarly
\[ (u + 1)(y + 1)(z + 1) = 10 \quad \text{...(2)} \]
\[ (x + 1)(z + 1)(u + 1) = 10 \quad \text{...(3)} \]
\[ (y + 1)(u + 1)(x + 1) = 10 \quad \text{...(4)} \]
Multiply
\[ (x + 1)^3(y + 1)^3(z + 1)^3(u + 1)^3 = 8000 \]
\[ (x + 1)(y + 1)(z + 1)(u + 1) = 20 \quad \text{...(5)} \]
So \[ \frac{\text{Eq.}(5)}{\text{Eq.(1)}} \Rightarrow u + 1 = \frac{20}{8} \Rightarrow u + 1 = \frac{5}{2} \Rightarrow u = \frac{3}{2} \]
\[ x + 1 = \frac{20}{10} \Rightarrow x + 1 = 2 \Rightarrow x = 1 \]
\[ y = 1 \]
\[ z = 1 \]

Q.2 If \(x,y,z,p,q,r\) are distinct real numbers such that
\[ \frac{1}{x + p} + \frac{1}{y + p} + \frac{1}{z + p} = \frac{1}{p} \]
\[ \frac{1}{x + q} + \frac{1}{y + q} + \frac{1}{z + q} = \frac{1}{q} \]
\[ \frac{1}{x + r} + \frac{1}{y + r} + \frac{1}{z + r} = \frac{1}{r} \]
find the numerical value of \[ \left( \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \right) \]

Sol.
\[ \frac{1}{x + p} + \frac{1}{y + p} + \frac{1}{z + p} = \frac{1}{p} \]
Let \( t = \frac{1}{p} \)
\[ \therefore \frac{1}{x + \frac{1}{t}} + \frac{1}{y + \frac{1}{t}} + \frac{1}{z + \frac{1}{t}} = t \]
\[ \frac{t}{xt + 1} + \frac{t}{yt + 1} + \frac{t}{zt + 1} = t \]
\[ \Rightarrow (tx + 1)(ty + 1) + (tz + 1)(tx + 1) + (tz + 1)(ty + 1) = (tx + 1)(ty + 1)(tz + 1) \]
Now this cubic equation has roots \( \frac{1}{p}, \frac{1}{q}, \frac{1}{r} \)
\[ \Rightarrow \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = -\text{coeff. of } t^2 \]
\[ \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \text{coeff. of } t^3 \]
Solving equation we get coefficient of $t^2 = 0$

$$\Rightarrow \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 0.$$

Q.3
ADC and ABC are triangles such that AD = DC and CA = AB. If $\angle CAB = 20^\circ$ and $\angle ADC = 100^\circ$, without using trigonometry, prove that $AB = BC + CD$.

**Sol.**

![Diagram of triangles ADC and ABC]

Extend BC to E such that CE = CD
Now CED is an equilateral triangle join AE
Let $\angle DAE = x$
Then $x = \angle DEA$
$\angle AEC = 60 - x$
$\angle EAC = 40 - x$
$\angle EAB = 60 - x$

$\triangle ABE$ is an isosceles triangle
$AB = BE$
$AB = BC + CE$
$AB = BC + CD$

Q.4
(a) $a$, $b$, $c$, $d$ are positive real numbers such that $abcd = 1$. Prove that

$$\frac{1+ab}{1+a} + \frac{1+bc}{1+b} + \frac{1+cd}{1+c} + \frac{1+da}{1+d} \geq 4$$

(b) In a scalene triangle ABC, $\angle BAC = 120^\circ$. The bisectors of the angles A, B and C meet the opposite sides in P, Q and R respectively. Prove that the circle on QR as diameter passes through the point P.

**Sol.**

(a) $\frac{1+ab}{1+a} = \frac{abcd + ab}{abcd + a} = \frac{bcd + b}{bcd + 1} = 1 + \frac{b-1}{bcd + 1}$

We have to prove

$$\sum \frac{b-1}{bcd + 1} \geq 0$$

Now
\[ \sum \frac{(b - 1)^2}{(bcd + 1)(b - 1)} \geq \frac{(a + b + c + d - 4)^2}{\Sigma(bcd + 1)(b - 1)} \]

By Titu's lemma (extended Cauchy)

Now let the expression \( \Sigma(bcd + 1)(b - 1) \) be \( E \)

\[
E = \sum \left( \frac{1}{a} + 1 \right)(b - 1) = \sum \left[ \frac{b}{a} + b - \frac{1}{a} - 1 \right]
\]

\[
= \left( \frac{b}{a} + \frac{c}{b} + \frac{d}{c} + \frac{1}{d} \right) - \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) = (a + b + c + d - 4)
\]

\[
= \left( \frac{b - 1}{a} + \frac{c - 1}{b} + \frac{d - 1}{c} + \frac{a - 1}{d} \right) = (a + b + c + d - 4)
\]

\[ \text{a + b + c + d} \geq 4 \text{ by AM – GM inequality} \]

\[ \text{ab} + \text{bc} + \text{cd} + \text{da} \geq 4 \text{ by AM – GM inequality} \]

Again \( \sum \frac{(b - 1)^2}{a(b - 1)} \geq 0 \)

Hence \( E \geq 0 \)

\[ a + b + c + d \geq 4 \text{ by AM – GM} \]

Hence \( \sum \frac{(b - 1)^2}{(bcd + 1)(b - 1)} \geq 0 \)

(b)

Produce BA upto x and CA upto y

\[ \angle BAP = \angle CAP = 60^\circ \quad \text{(given)} \]

\[ \angle YAQ = \angle PAQ = 60^\circ \]

So AQ is bisector of ext. \( \angle A \) of \( \triangle ABP \)

BQ is bisector of interior \( \angle B \) of \( \triangle ABP \)

So Q is excentre of \( \triangle ABP \)

So PQ bisect \( \angle APC \)

Let \( \angle APQ = \angle CPQ = \alpha \) (say)

Similarly \( \angle BPA = \angle QPA = \beta \)

\[ \angle BPA + \angle CPA = 180^\circ \]

\[ 2\beta + 2\alpha = 180^\circ \]

\[ \alpha + \beta = 90^\circ \]

\[ \angle QPR = 90^\circ \]

Hence circle on QR as diameter passes through point P.
Q.5 (a) Prove that $x^4 + 3x^3 + 6x^2 + 9x + 12$ cannot be expressed as a product of two polynomials of degree 2 with integer coefficients.

(b) $2n + 1$ segments are marked on a line. Each of these segments intersects at least $n$ other segments. Prove that one of these segments intersects all other segments.

Sol. (a) Let $x^4 + 3x^3 + 6x^2 + 9x + 12$
\[= (x^2 + Ax + B)(x^2 + Cx + D)\]
\[= x^4 + Cx^3 + Dx^2 + Ax^3 + ACx^2 + ADx + Bx^2 + BCx + BD\]
\[= x^4 + (A + C)x^3 + (D + AC + B)x^2 + (AD + BC)x + BD\]

Compare coefficient

\[A + C = 3\]
\[B + D + AC = 6\]
\[AD + BC = 9\]
\[BD = 12\]

Case-I
\[B = 1, D = 12\]

\[\therefore A + C = 3\]

\[12A + C = 9\] has no integer solution

Case-II
\[B = -1, D = -12\]
\[C = +12, A = -9\]

\[C + A = 3\] have no integer solution

Case-III
\[B = 2, D = 6\]
\[2C + 6A = 9\]

\[C + A = 3\] have no integer solution

Case-IV
\[B = -2, D = -6\]
\[2C + 6A = -9\]

\[A + C = 3\] have no integer solution

So $x^4 + 3x^3 + 6x^2 + 9x + 12$ cannot be expressed as a product of two polynomial of degree 2 with integer coefficient.

(b) The question is to be done by induction

Let's take $k = 1$

⇒ There are three segments such that all segments intersect at least one segment

Only possibility is as follows

1 3

2

Here 2 intersect both (1) and (3)

Hence statement is true for $k = 1$
Q.6 If a, b, c, d are positive real numbers such that \( a^2 + b^2 = c^2 + d^2 \) and \( a^2 + d^2 - ad = b^2 + c^2 + bc \), find the value of \( \frac{ab + cd}{ad + bc} \).

Sol.

\( a^2 + b^2 = c^2 + d^2 \)

\[(a + b)^2 - (c - d)^2 = 2(ab + cd) \quad \text{...(1)}\]

\[(c + d)^2 - (a - b)^2 = 2(ab + cd) \quad \text{...(2)}\]

\[(1) \times (2)\]

\[4(ab + cd)^2 = (a + b + c - d)(a + b - c + d) \]

\[= (c + d + a - b)(c + d - a + b) \quad \text{...(3)}\]

Also

\[a^2 + d^2 - ad = b^2 + c^2 + bc\]

\[(a + d)^2 - (b - c)^2 = 3(ad + bc) \quad \text{...(4)}\]

\[(b + c)^2 - (a - d)^2 = (ad + bc) \quad \text{...(5)}\]

\[(4) \times (5)\]

\[3(ad + bc)^2 = (a + d + b - c)(a + d - b + c) \]

\[= (b + c + a - d)(b + c - a + d) \quad \text{...(6)}\]

RHS of (3) & (6) is equal

\[\Rightarrow 3(ad + bc)^2 = 4(ab + cd)^2\]

\[\Rightarrow \frac{3}{4} = \left(\frac{ab + cd}{ad + bc}\right)^2 \Rightarrow \frac{ab + cd}{ad + bc} = \frac{\sqrt{3}}{2}\]