MATHEMATICS

Paper & Solutions

Code : 65/2/N

Time : 3 Hrs. **Max. Marks : 100**

General Instruction :

(i) All questions are compulsory.

(ii) Please check that this question paper contains **26** questions.

(iii) Questions **1-6** in **Section-A** are very short-answer type questions carrying **1** mark each.

(iv) Questions **7-19** in **Section-B** are long answer **I** type questions carrying **4** marks each.

(v) Questions **20-26** in **Section-C** are long answer **II** type questions carrying **6** marks each.

(vi) Please write down the serial number of the question before attempting it.

SECTION - A

Question numbers 1 to 6 carry 1 mark each

1. If $\vec{a} = 4\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - 2\hat{j} + \hat{k}$, then find a unit vector parallel to the vector $\vec{a} + \vec{b}$.

Sol. $\vec{a} + \vec{b} = 4\hat{i} - \hat{j} + \hat{k} + 2\hat{i} - 2\hat{j} + \hat{k}$

$= 6\hat{i} - 3\hat{j} + 2\hat{k}$

unit vector parallel to $\vec{a} + \vec{b} = \dfrac{(6\hat{i} - 3\hat{j} + 2\hat{k})}{\sqrt{36+9+4}} = \dfrac{(6\hat{i} - 3\hat{j} + 2\hat{k})}{7}$

2. Find λ and μ if

$(\hat{i} + 3\hat{j} + 9\hat{k}) \times (3\hat{i} - \lambda\hat{j} + \mu\hat{k}) = \vec{0}$.

Sol. $(\hat{i} + 3\hat{j} + 9\hat{k}) \times (3\hat{i} - \lambda\hat{j} + \mu\hat{k}) = \vec{0}$.

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 9 \\ 3 & -\lambda & \mu \end{vmatrix} = \vec{0}$$

$\Rightarrow \hat{i}(3\mu + 9\lambda) - \hat{j}(\mu - 27) + \hat{k}(-\lambda - 9) = 0\hat{i} + 0\hat{j} + 0\hat{k}$

$\Rightarrow 3\mu + 9\lambda = 0$

$\Rightarrow \mu - 27 = 0$

$\Rightarrow \boxed{\mu = 27}$

(i) to $3 \times 27 + 9\lambda = 0$

$\Rightarrow \boxed{\lambda = -9}$

3. Write the sum of intercepts cut off by the plane $\vec{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) - 5 = 0$ on the three axes.

Sol. Plane $\vec{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) - 5 = 0$

$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} + \hat{j} - \hat{k}) - 5 = 0$

$\Rightarrow 2x + y - z = 5$

$\Rightarrow \dfrac{x}{(5/2)} + \dfrac{y}{(5)} + \dfrac{z}{(-5)} = 1$

$\Rightarrow$ sum of intercepts $= \dfrac{5}{2} + 5 - 5 = \boxed{\dfrac{5}{2}}$

4. For what values of k, the system of linear equations
$x + y + z = 2$
$2x + y - z = 3$
$3x + 2y + kz = 4$
has a unique solution ?

Sol. $x + y + z = 2$ (i)
$2x + y - z = 3$ (ii)
$3x + 2y + kz = 4$ (iii)
The system of linear equation has unique solution then

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{vmatrix} \neq 0$$

$\Rightarrow 1\,(k + 2) - 1\,(2k + 3) + 1\,(4 - 3) \neq 0$
$\Rightarrow k + 2 - 2k - 3 + 1 \neq 0$
$\Rightarrow k \neq 0$
So values of $k = R - \{0\}$

5. If A is a 3 × 3 matrix and $|3A| = k|A|$, then write the value of k.

Sol. A is a matrix of 3 ×3 say

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$3B = \begin{vmatrix} 3a_{11} & 3a_{12} & 3a_{13} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ 3a_{31} & 3a_{32} & 3a_{33} \end{vmatrix}$$

$\therefore |3A| = 3 \times 3 \times 3\,|A| = 27\,|A|$
Which is given as $k\,|A|$ so $\boxed{k = 27}$

6. If $A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$, find α satisfying $0 < \alpha < \frac{\pi}{2}$ when $A + A^T = \sqrt{2}I_2$; where A^T is transpose of A.

Sol. $A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$

$A^T = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$

$$\Rightarrow A + A^T = \begin{pmatrix} \cos\alpha + \cos\alpha & 0 \\ 0 & 2\cos\alpha \end{pmatrix} = \begin{pmatrix} 2\cos\alpha & 0 \\ 0 & 2\cos\alpha \end{pmatrix}$$

$\because A + A^T = \sqrt{2}\,I_2$

$$\text{So } \begin{pmatrix} 2\cos\alpha & 0 \\ 0 & 2\cos\alpha \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$$

by comparing : $-2\cos\alpha = \sqrt{2}$

$$\cos\alpha = \frac{1}{\sqrt{2}}$$

$$\text{so} \quad \alpha = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$$

SECTION - B

Question numbers 7 to 19 carry 4 marks each

7. A bag X contains 4 white balls and 2 black balls, while another bag Y contains 3 white balls and 3 black balls. Two balls are drawn (without replacement) at random from one of the bags and were found to be one white and one black. Find the probability that the balls were drawn from bag Y.

OR

A and B throw a pair of dice alternately, till one of them gets a total of 10 and wins the game. Find their respective probabilities of winning, if A starts first.

Sol.

X		Y	
4 White	2 Black	3White	3Black

Event $E_1 \to$ Bag X is selected

Event $E_2 \to$ Bag Y is selected

Event A $\to$ 1 white 1 black is taken out

$P(E_1) = \frac{1}{2}$, $P(E_2) = \frac{1}{2}$, $P(A/E_1) = \frac{{}^4C_1 \cdot {}^2C_1}{{}^6C_2}$, $P(A/E_2) = \frac{{}^3C_1 \cdot {}^3C_1}{{}^6C_2}$

$P(A) = P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2)$

$$= \frac{1}{2}\cdot\frac{4_{C_1}\cdot 2_{C_1}}{6_{C_2}} + \frac{1}{2}\frac{3_{C_1}\cdot 3_{C_1}}{6_{C_2}}$$

probability balls are drawn from bag 1 : $P(E_2/A) = \frac{P(E_2)P(A/E_2)}{P(A)}$ (Using Baye's Theorem)

$$\Rightarrow \frac{\frac{1}{2}\cdot\frac{3_{C_1}\cdot 3_{C_1}}{6_{C_2}}}{\frac{1}{2}\cdot\frac{4_{C_1}\cdot 2_{C_1}}{6_{C_2}} + \frac{1}{2}\frac{3_{C_1}\cdot 3_{C_1}}{6_{C_2}}} = \frac{9}{4\cdot 2+9} = \frac{9}{17}$$

OR

total of 10 : (6, 4) (4, 6) (5, 5)

$$\Rightarrow \quad P = \frac{3}{36} = \frac{1}{12}$$

A starts first $\quad \overset{\checkmark}{A} \quad \overset{\times}{A}\ \overset{\times}{B}\ \overset{\checkmark}{A} \quad \overset{\times}{A}\ \overset{\times}{B}\ \overset{\times}{A}\ \overset{\times}{B}\ \overset{\checkmark}{A}$

$$P(\text{A wins}) = \frac{1}{12} + \frac{11}{12}\cdot\frac{11}{12}\cdot\frac{1}{12} + \frac{11}{12}\cdot\frac{11}{12}\cdot\frac{11}{12}\cdot\frac{11}{12}\cdot\frac{1}{12} + \ldots\ldots$$

$$= \frac{1}{12}\left(1 + \left(\frac{11}{12}\right)^2 + \left(\frac{11}{12}\right)^4 + \ldots\right)$$

$$= \frac{1}{12}\left(\frac{1}{1-\left(\frac{11}{12}\right)^2}\right) = \frac{1}{12}\cdot\frac{144}{(144-121)}$$

$P(\text{A wins}) = \frac{12}{23}$

$P(\text{B wins}) = 1 - P(\text{A wins}) = 1 - \frac{12}{23} = \frac{11}{23}$.

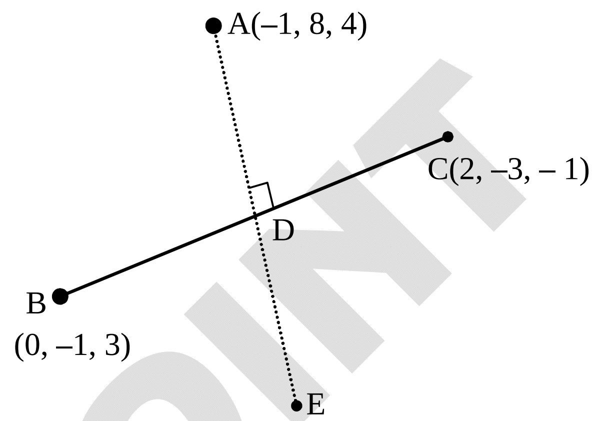
8. Find the coordinates of the foot of perpendicular drawn from the point A(–1, 8, 4) to the line joining the points B(0, –1, 3) and C(2, –3, –1). Hence find the image of the point A in the line BC.

Sol. Equation of BC

$$\frac{x-0}{2-0} = \frac{y+1}{-3+1} = \frac{z-3}{-1-3}$$

$$\Rightarrow \frac{x}{2} = \frac{y+1}{-2} = \frac{z-3}{-4}$$

$$\Rightarrow \frac{x}{1} = \frac{y+1}{-1} = \frac{z-3}{-2} = \lambda \text{ (say)}$$



$\Rightarrow$ D(λ, – λ – 1, –2λ + 3)

DR's of AD

$\Rightarrow \lambda + 1, -\lambda - 9, -2\lambda + 3 - 4$

$\Rightarrow \lambda + 1, -\lambda - 9, -2\lambda - 1$

$\because$ AD $\perp$ BC

$\Rightarrow (\lambda + 1)(1) + (-\lambda - 9)(-1) + (-2\lambda - 1)(-2) = 0$

$\Rightarrow \lambda + 1 + \lambda + 9 + 4\lambda + 2 = 0$

$\Rightarrow 6\lambda + 12 = 0 \Rightarrow \lambda = -2$

foot of perpendicular to D(–2, 1, 7)

let image of A w.r.to line Bc is E(x_1, y_1, z_1)

mid point of AE $\left(\frac{x_1-1}{2}, \frac{y_1+8}{2}, \frac{z_1+4}{2}\right) \equiv (-2, 1, 7)$

$x_1 - 1 = -4$	$y_1 + 8 = 2$	$z_1 + 4 = 14$
$(x_1 = -3)$	$(y_1 = -6)$	$(z_1 = 10)$

E (–3, –6, 10) Ans.

9. Show that the four points A(4, 5, 1), B(0, –1, –1), C(3, 9, 4) and D(–4, 4, 4) are coplanar.

Sol. $\overrightarrow{AB} = (0-4)\hat{i} + (-1-5)\hat{j} + (-1-1)\hat{k}$

$\Rightarrow \overrightarrow{AB} = -4\hat{i} - 6\hat{j} - 2\hat{k}$

$\Rightarrow \overrightarrow{AC} = (3-4)\hat{i} + (9-5)\hat{j} + (4-1)\hat{k}$

$= -\hat{i} + 4\hat{j} + 3\hat{k}$

$\Rightarrow \overrightarrow{AD} = (-4-4)\hat{i} + (4-5)\hat{j} + (4-1)\hat{k}$

$= -8\hat{i} - \hat{j} + 3\hat{k}$

We know that three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar if $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

$$\Rightarrow [\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}] = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix} = -4(12+3) + 6(-3+24) - 2(1+32)$$

$$= -60 + 126 - 166 = 0$$

$\therefore$ A,B,C,D are coplanar.

10. Find the particular solution of the differential equation

$2y\,e^{x/y}\,dx + (y - 2x\,e^{x/y})dy = 0$

given that x = 0 when y = 1.

Sol. $2ye^{x/y}dx + (y - 2xe^{x/y})dy = 0$

$$\Rightarrow 2e^{x/y} = -\left(1 - \frac{2x}{y}e^{x/y}\right)\frac{dy}{dx}$$

put $\frac{x}{y} = t$

$x = yt$

$$1 = \frac{dy}{dx}t + y\frac{dt}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{t}\left(1 - y\frac{dt}{dx}\right)$$

$$\Rightarrow 2e^t = -(1 - 2te^t)\frac{1}{t}\left(1 - \frac{ydt}{dx}\right)$$

$$\Rightarrow 2e^t = -(1 - 2te^t)\frac{1}{t}\left(1 - y\frac{dt}{dx}\right)$$

$$\Rightarrow 2te^t = (2te^t - 1)\left(1 - y\frac{dt}{dx}\right) = 2te^t - 1 - y2te^t\frac{dt}{dx} + \frac{ydt}{dx}$$

$$\Rightarrow 1 = y(1 - 2te^t)\frac{dt}{dx} = \frac{x}{t}(1 - 2te^t)\frac{dt}{dx}$$

$$\int\frac{dx}{x} = \int\left(\frac{1}{t} - 2e^t\right)dt$$

$\Rightarrow \ell nx = \ell nt - 2e^t + c$

$$\Rightarrow \ell nx = \ell n\left(\frac{x}{y}\right) - 2e^{x/y} + c$$

$\Rightarrow \ell ny = -2e^{x/y} + c$

At x = 0, y = 1

$\Rightarrow 0 = -2e^0 + c \Rightarrow c = 2$

$\boxed{\ell ny = 2 - 2e^{x/y}}$

11. Find the particular solution of differential equation: $\frac{dy}{dx} = -\frac{x + y\cos x}{1 + \sin x}$ given that y = 1 when x = 0.

Sol. Given differential equation

$$\frac{dy}{dx} = -\frac{x + y\cos x}{1 + \sin x}$$

$$\Rightarrow \frac{dy}{dx} + \frac{\cos x}{1 + \sin x}y = \frac{-x}{1 + \sin x}$$

Integrating factor = $e^{\int\left(\frac{\cos x}{1+\sin x}\right)dx}$

$= e^{\log(1 + \sin x)} = 1 + \sin x$

$$y\cdot(1+\sin x) = \int \frac{-x}{(1+\sin x)}(1+\sin x)\,dx + c$$

$$\Rightarrow\ y\cdot(1+\sin x) = -\frac{x^2}{2} + c$$

At $x = 0$, $y = 1$

$1\cdot(1+0) = c \ \Rightarrow\ c = 1$

So $\boxed{y\,(1+\sin x) = -\frac{x^2}{2} + 1}$

12. Find : $\int (x+3)\sqrt{3-4x-x^2}\,dx.$

Sol. $\int (x+3)\sqrt{3-4x-x^2}\,dx$

Set $\Rightarrow$ $x + 3 = p\cdot\frac{d}{dx}(3-4x-x^2) + q$

$\Rightarrow x + 3 = p\cdot(-4-2x) + q$

so $-2p = 1 \Rightarrow p = -\frac{1}{2}$

$-4p + q = 3 \Rightarrow q = 1$

so given integral

$$\int\left\{-\frac{1}{2}\cdot(-4-2x)+1\right\}\sqrt{3-4x-x^2}\,dx$$

$$\Rightarrow\ \int -\frac{1}{2}\cdot(-4-2x)\sqrt{3-4x-x^2}\,dx + \int\sqrt{3-4x-x^2}\,dx$$

↑ Make perfect sq.

put $3 - 4x - x^2 = t$

$\Rightarrow (-4-2x)dx = dt$

$$-\frac{1}{2}\cdot\int\sqrt{t}\,dt + \int\sqrt{7-(x+2)^2}\,dx$$

$$\Rightarrow -\frac{1}{2}\cdot\frac{2}{3}\cdot t^{3/2} + \frac{(x+2)}{2}\sqrt{7-(x+2)^2} + \frac{7}{2}\cdot\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right) + c$$

$$\Rightarrow -\frac{1}{3}\cdot(3-4x-x^2)^{3/2} + \left(\frac{x+2}{2}\right)\sqrt{3-4x-x^2} + \frac{7}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right) + c$$

13. Find: $\int \frac{(2x-5)e^{2x}}{(2x-3)^3}\,dx$

OR

Find: $\int \frac{x^2+x+1}{(x^2+1)\,(x+2)}\,dx$

Sol. $\int \frac{(2x-5)e^{2x}}{(2x-3)^3}dx$ put 2x = t

$\Rightarrow \frac{1}{2}\cdot\int \frac{(t-5)e^t}{(t-3)^3}dt$

$\Rightarrow \frac{1}{2}\cdot\int e^t\left(\frac{t-3}{(t-3)^3}-\frac{2}{(t-3)^3}\right)dt$

$\Rightarrow \frac{1}{2}\cdot\int e^t\left(\frac{1}{(t-3)^2}-\frac{2}{(t-3)^3}\right)dt$

f(t) f '(t)

$\because \int e^x(f(x)+f'(x))dx$

$= e^x f(x) + c$

so $\frac{1}{2}\cdot e^t\cdot\frac{1}{(t-3)^2}+c$

$\Rightarrow \frac{1}{2}\cdot e^{2x}\frac{1}{(2x-3)^2}+c$

OR

$\int \frac{x^2+x+1}{(x^2+1)(x+2)}dx$

$\int \frac{x^2+x+1}{(x^2+1)(x+2)}dx = \int\left(\frac{Ax+B}{(x^2+1)}+\frac{C}{(x+2)}\right)dx$

$\Rightarrow x^2 + x + 1 = (Ax + B)(x + 2) + C(x^2 + 1)$

$\Rightarrow x^2 + x + 1 = Ax^2 + 2Ax + Bx + 2B + Cx^2 + C$

A + C = 1 ....(i)

2A + B = 1 ....(ii)

2B + C = 1 ....(iii)

by equation (i), (ii) and (iii) we get

$\boxed{C = 3/5}$

$\therefore\ B = \frac{6}{5}-1 \Rightarrow \boxed{B = \frac{1}{5}}$

$\therefore\ \boxed{A = 1-\frac{3}{5}=\frac{2}{5}}$

$\therefore\ \int\left(\frac{1}{5}\cdot\frac{(2x+1)}{(x^2+1)}+\frac{3}{5}\cdot\frac{1}{(x+2)}\right)dx$

$= \frac{1}{5}\int\frac{2x}{x^2+1}dx+\frac{1}{5}\cdot\int\frac{1}{x^2+1}dx+\frac{3}{5}\int\frac{1}{x+2}dx$

$= \frac{1}{5}\cdot\log(x^2+1)+\frac{1}{5}\cdot\tan^{-1}x+\frac{3}{5}.\log|x+2|+c$

14. Find the equation of tangents to the curve $y = x^3 + 2x - 4$, which are perpendicular to line $x + 14y + 3 = 0$.

Sol. $x + 14y + 3 = 0$

$m = -\frac{1}{14}$

slope of perpendicular line = 14.

curve $y = x^3 + 2x - 4$

$\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 3x_1^2 + 2 = 14$

$3x_1^2 = 12$

$x_1^2 = 4$

$x_1 = \pm 2$

$x_1 = 2$ $y_1 = 8 + 4 - 4 = 8$ point (2, 8)

$x_1 = -2$ $y_1 = -8 - 4 - 4 = -16$ point (–2, –16)

tangent at (2, 8) $y - 8 = 14(x - 2) = 14x - 28$, $\boxed{y = 14x - 20}$

and at (–2, –16) $y + 16 = 14(x + 2)$

$y + 16 = 14x + 28$, $\boxed{y = 14x + 12}$

15. If $x\cos(a + y) = \cos y$ then prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$

Hence show that $\sin a \frac{d^2y}{dx^2} + \sin 2(a + y) \frac{dy}{dx} = 0$.

OR

Find $\frac{dy}{dx}$ if $y = \sin^{-1}\left[\frac{6x - 4\sqrt{1-4x^2}}{5}\right]$

Sol. $x\cos(a + y) = \cos y$

$x = \frac{\cos y}{\cos(a+y)}$

Differentiate with respect to 'x'

$$1 = \frac{\cos(a+y)\left(-\sin y\frac{dy}{dx}\right) - \cos y\left(-\sin(a+y)\frac{dy}{dx}\right)}{\cos^2(a+y)}$$

$$1 = \frac{(\sin(a+y)\cos y - \cos(a+y)\sin y)\frac{dy}{dx}}{\cos^2(a+y)}$$

$\cos^2(a + y) = \sin(a + y - y)\ \frac{dy}{dx}$

$$\boxed{\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}}$$

and

$\sin a\ \frac{dy}{dx} = \frac{1+\cos 2(a+y)}{2}$

Again, Differentiate with respect to 'x'

$\sin a\ \frac{d^2y}{dx^2} = 0 - \frac{\sin 2(a+y)}{2}\cdot 2\frac{dy}{dx}$

$$\boxed{\sin a\frac{d^2y}{dx^2} + \sin 2(a+y)\frac{dy}{dx} = 0}$$

OR

$y = \sin^{-1}\left[\frac{6x - 4\sqrt{1-4x^2}}{5}\right]$

$= \sin^{-1}\left[\frac{3}{5}\cdot 2x - \frac{4}{5}\sqrt{1-4x^2}\right]$

$= \sin^{-1}\left[2x\cdot\sqrt{1-\left(\frac{4}{5}\right)^2} - \frac{4}{5}\sqrt{1-(2x)^2}\right]$

$y = \sin^{-1}2x - \sin^{-1}\ \frac{4}{5}$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-(2x)^2}}\cdot 2 - 0 = \frac{2}{\sqrt{1-4x^2}}$

16. Evaluate : $\int_{-2}^{2} \frac{x^2}{1+5^x}dx$.

Sol. $I = \int_{-2}^{2}\frac{x^2}{1+5^x}dx = \int_{0}^{2}\left(\frac{x^2}{1+5^x} + \frac{(-x)^2}{1+5^{-x}}\right)dx$

$= \int_{0}^{2}\left(\frac{x^2}{1+5^x} + \frac{x^2 5^x}{5^x+1}\right)dx = \int_{0}^{2} x^2\frac{(1+5^x)}{5^x+1}dx$

$= \int_{0}^{2} x^2 dx = \left.\frac{x^3}{3}\right|_{0}^{2} = \frac{8}{3} - 0 = \frac{8}{3}$

17. If $f(x)=\begin{cases}\dfrac{\sin(a+1)x+2\sin x}{x}, & x<0\\ 2, & x=0\\ \dfrac{\sqrt{1+bx}-1}{x}, & x>0\end{cases}$

is continuous at x = 0, then find the values of a and b.

Sol. $f(x)=\begin{cases}\dfrac{\sin(a+1)x+2\sin x}{x}, & x<0\\ 2, & x=0\\ \dfrac{\sqrt{1+bx}-1}{x}, & x>0\end{cases}$

value at x = 0 is 2

LHL : $\lim\limits_{h\to 0} f(0-h)$

$= \lim\limits_{h\to 0}\dfrac{-\sin(a+1)h-2\sin h}{-h}$

$= \lim\limits_{h\to 0}\left(\dfrac{\sin(a+1)h}{h}+\dfrac{2\sin h}{h}\right)$ $\left(\because \lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1\right)$

= (a + 1) + 2 = a + 3

RHL : $\lim\limits_{h\to 0} f(0+h)$

$= \lim\limits_{h\to 0}\dfrac{\left(\sqrt{1+bh}-1\right)}{h}\dfrac{\left(\sqrt{1+bh}+1\right)}{\left(\sqrt{1+bh}+1\right)} = \dfrac{b}{2}$(rationalize)

since it is continuous

$\therefore$ LHL = RHL = f(0)

$\therefore$ a + 3 = $\dfrac{b}{2}$ = 2

$\therefore$ $\boxed{a = -1, b = 4}$

18. A typist charges ₹ 145 for typing 10 English and 3 Hindi pages, while charges for typing 3 English and 10 Hindi pages are ₹ 180. Using matrices, find the charges of typing one English and one Hindi page separately. However typist charged only ₹ 2 per page from a poor student Shyam for 5 Hindi pages. How much less was charged from this poor boy ? Which values are reflected in this problem ?

Sol. Charges of typing one English page = Rs x

Charges of typing one Hindi page = Rs y

$10x + 3y = 145$

$3x + 10y = 180$

$$\begin{bmatrix} 10 & 3 \\ 3 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 145 \\ 180 \end{bmatrix}$$

$Ax = B$

$A^{-1}Ax = A^{-1}B$

$Ix = \frac{adjA}{|A|}B \quad |A| = 100 - 9 = 91$

$$x = \frac{1}{91}\begin{bmatrix} 10 & -3 \\ -3 & 10 \end{bmatrix}\begin{bmatrix} 145 \\ 180 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{91}\begin{bmatrix} 1450-540 \\ -435+1800 \end{bmatrix} = \begin{bmatrix} 910 \\ 1365 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$$

$x = 10$

$y = 15$

Poor student he charged = 2× 5 = Rs 10

actual cost = 15×5 = Rs 75

less charged = 75 – 10 = Rs 65 ₹

This problem reflect on human values "Kindness" etc.

19. Solve for x : $\tan^{-1}(x-1) + \tan^{-1}x + \tan^{-1}(x+1) = \tan^{-1}3x$

OR

Prove that : $\tan^{-1}\left(\frac{6x-8x^3}{1-12x^2}\right) - \tan^{-1}\left(\frac{4x}{1-4x^2}\right) = \tan^{-1}2x\,;\ |2x| < \frac{1}{\sqrt{3}}$

Sol. $\tan^{-1}(x-1) + \tan^{-1}(x+1) = \tan^{-1}3x - \tan^{-1}x$

$\Rightarrow \tan^{-1}\left(\frac{x-1+x+1}{1-(x-1)(x+1)}\right) \quad \tan^{-1}\left(\frac{3x-x}{1+3x^2}\right)$

$\Rightarrow \frac{2x}{1-(x^2-1)} = \frac{2x}{1+3x^2}$

$\Rightarrow 2x(1+3x^2) = 2x(2-x^2)$

$\Rightarrow 2x[(1+3x^2) - 2 + x^2] = 0$

$\Rightarrow x(4x^2-1) = 0$

$\Rightarrow x = 0 \ x^2 = \frac{1}{4}$

$\Rightarrow x = \pm\frac{1}{2}$

$\Rightarrow x = 0, +\frac{1}{2}, -\frac{1}{2}$

OR

$$\tan^{-1}\left(\frac{3(2x)-(2x)^3}{1-3(2x)^2}\right)-\tan^{-1}\left(\frac{2(2x)}{1-(2x)^2}\right)$$

let $2x = \tan\theta$ $\quad |2x| \le \frac{1}{\sqrt{3}}$

$$\tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)-\tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$

$\Rightarrow \tan^{-1}(\tan 3\theta) - \tan^{-1}(\tan 2\theta)$

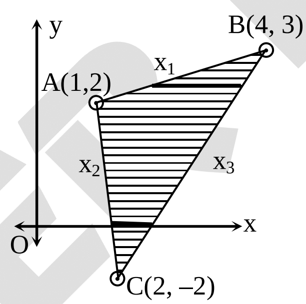
$\Rightarrow 3\theta - 2\theta = \theta = \tan^{-1} 2x$ H.P.

SECTION - C

Question number 20 to 26 carry 6 mark each

20. Using the method of integration, find the area of the triangular region whose vertices are (2, –2), (4, 3) and (1, 2).

Sol.



equation of line AB :

$$y-2=\left(\frac{3-2}{4-1}\right)(x-1)$$

$$y-2=\frac{1}{3}(x-1)$$

$\Rightarrow x = 3y - 5$ (a) (line x_1)

equation of line AC :

$$y-2=\left(\frac{-2-2}{2-1}\right)(x-1)$$

$\Rightarrow y - 2 = -4x + 4$

$\Rightarrow x = \frac{6-y}{4}$ (b) (line x_2)

equation of line BC :

$$y+2=\left(\frac{3+2}{4-2}\right)(x-2)$$

$2y + 4 = 5x - 10$

$$x=\frac{2y+14}{5} \qquad \text{(c) (line } x_3\text{)}$$

Area of ΔABC

$$\Rightarrow \int_{-2}^{2}(x_3-x_2)dy+\int_{2}^{3}(x_3-x_1)dy$$

$$\Rightarrow \int_{-2}^{2}\left(\frac{2y+14}{5}-\frac{6-y}{4}\right)dy+\int_{2}^{3}\left(\frac{2y+14}{5}-(3y-5)\right)dy$$

$$\Rightarrow \frac{13}{2}\text{ sq. units}$$

21. Using properties of determinants, prove that

$$\begin{vmatrix}(x+y)^2 & zx & zy\\ zx & (z+y)^2 & xy\\ zy & xy & (z+x)^2\end{vmatrix}=2xyz(x+y+z)^3$$

OR

If $A=\begin{pmatrix}1 & 0 & 2\\ 0 & 2 & 1\\ 2 & 0 & 3\end{pmatrix}$ and $A^3 - 6A^2 + 7A + kI_3 = O$ find k.

Sol.
$$\begin{vmatrix}(x+y)^2 & zx & zy\\ zx & (z+y)^2 & xy\\ zy & xy & (z+x)^2\end{vmatrix}$$

$R_1 \to zR_1$, $R_2 \to xR_2$, $R_3 \to yR_3$ we get

$$\frac{1}{xyz}\begin{vmatrix}z(x+y)^2 & z^2x & z^2y\\ zx^2 & x(z+y)^2 & x^2y\\ zy^2 & xy^2 & y(z+x)^2\end{vmatrix}$$

Taking z, x, y common from, C_1, C_2, C_3 respectively, we get

$$\frac{xyz}{xyz}\begin{vmatrix}(x+y)^2 & z^2 & z^2\\ x^2 & (z+y)^2 & x^2\\ y^2 & y^2 & (z+x)^2\end{vmatrix}$$

$C_1 \to C_1 - C_3$, $C_2 \to C_2 - C_3$

$$\begin{vmatrix}(x+y)^2-z^2 & 0 & z^2\\ 0 & (z+y)^2-x^2 & x^2\\ y^2-(z+x)^2 & y^2-(z+x)^2 & (z+x)^2\end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} (x+y+z)(x+y-z) & 0 & z^2 \\ 0 & (z+y-x)(z+y+x) & x^2 \\ (y-z-x)(y+z+x) & (y-z-x)(y+z+x) & (z+x)^2 \end{vmatrix}$$

$$\Rightarrow (x+y+z)^2 \begin{vmatrix} x+y-z & 0 & z^2 \\ 0 & z+y-x & x^2 \\ y-z-x & y-z-x & (z+x)^2 \end{vmatrix}$$

$R_3 \to R_3 - R_2 - R_1$

$$\Rightarrow (x+y+z)^2 \begin{vmatrix} x+y-z & 0 & z^2 \\ 0 & z+y-x & x^2 \\ -2x & -2z & 2zx \end{vmatrix}$$

$C_1 \to C_1 + \frac{1}{z} C_3,\ C_2 \to C_2 + \frac{1}{x} C_3$

$$\Rightarrow (x+y+z)^2 \begin{vmatrix} x+y & \frac{z^2}{x} & z^2 \\ \frac{x^2}{z} & z+y & x^2 \\ 0 & 0 & 2zx \end{vmatrix}$$

Expanding along R_3

$$\Rightarrow 2x\,z\,(x+y+z)^2 \left((x+y)(z+y) - \frac{x^2}{z} \cdot \frac{z^2}{x} \right)$$

$$\Rightarrow 2x\,z\,(x+y+z)^2\,(xz + x\,y + yz + y^2 - xz)$$

$$\Rightarrow 2xyz\,(x+y+z)^3 \text{ H.P.}$$

OR

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \Rightarrow A^2 = A.A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+4 & 0+0+0 & 2+0+6 \\ 0+0+2 & 0+4+0 & 0+2+3 \\ 2+0+6 & 0+0+0 & 4+0+9 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5+0+16 & 0+0+0 & 10+0+24 \\ 2+0+10 & 0+8+0 & 4+4+15 \\ 8+0+26 & 0+0+0 & 16+0+39 \end{bmatrix} = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$$

Taking $A^3 - 6A^2 + 7A + kI_3 = 0$

$$\begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + kI_3 = 0$$

Or $$\begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - \begin{bmatrix} 30 & 0 & 48 \\ 12 & 24 & 30 \\ 48 & 0 & 78 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 14 \\ 0 & 14 & 7 \\ 14 & 0 & 21 \end{bmatrix} + kI_3 = 0$$

Or $$\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} + \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow k - 2 = 0$

$\Rightarrow \boxed{k = 2}$

22. A retired person wants to invest an amount of ₹ 50,000. His broker recommends investing in two type of bonds 'A' and 'B' yielding 10% and 9% return respectively on the invested amount . He decides to invest at least ₹ 20,000 in bond 'A' and at least ₹ 10,000 in bond 'B'. He also wants to invest at least as much in bond 'A' as in bond 'B'. Solve this linear programming problem graphically to maximise his returns.

Sol. Let Rs 'x' invest in bond A and Rs 'y' invest in bond B.

then A.T.P.

Maxmise $z = \frac{10}{100}x + \frac{9}{100}y$ ...(1)

subject to constraints

$x + y \le 50{,}000$ ...(a)

$x \ge 20{,}000$ ...(b)

$y \ge 10{,}000$ ...(c)

and $x \ge y$

or $x - y \ge 0$ ...(d)

and $x \ge 0, y \ge 0$

Now change inequality into equations

$x + y = 50{,}000$, $x = 20{,}000$, $y = 10{,}000$ and $x = y$

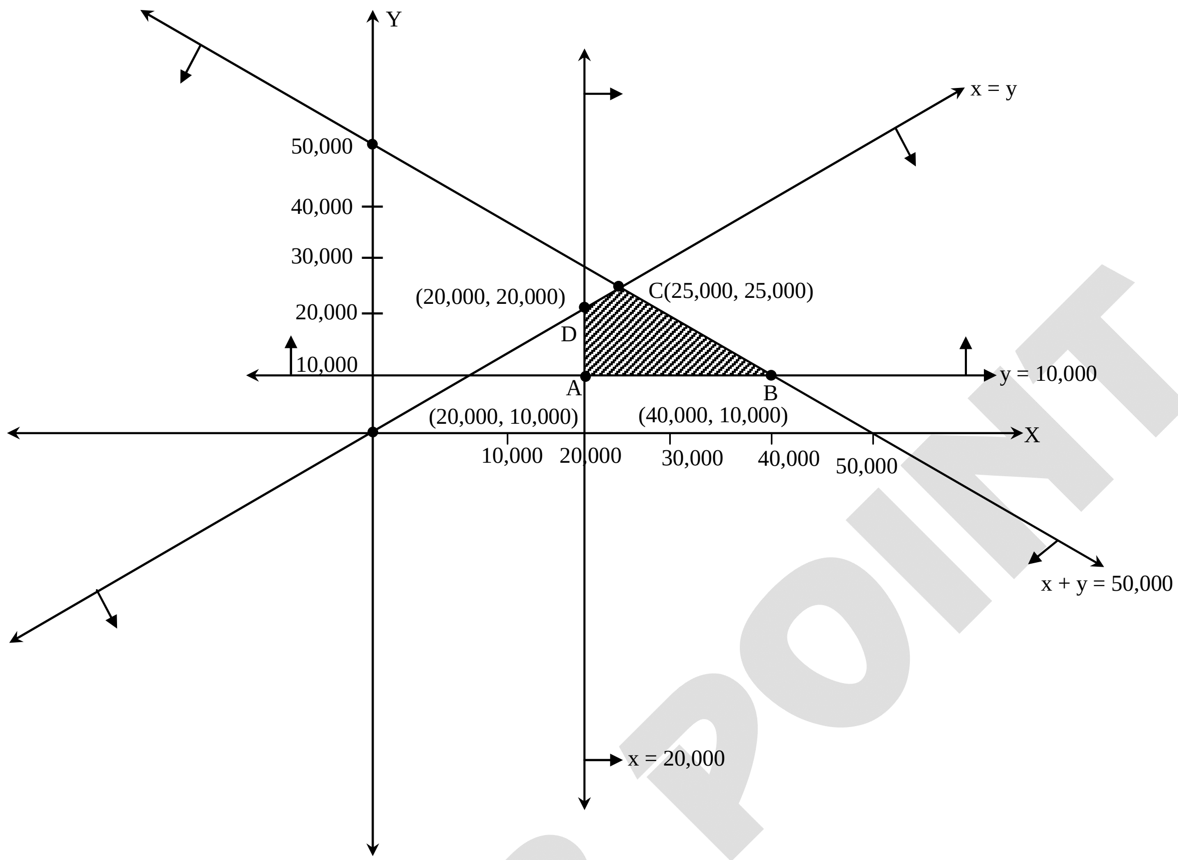
x	0	50,000
y	50,000	0

Region: put (0, 0) in (a), (b), (c), (d)

$0 \le 50{,}000$ (towards origin)

$0 \ge 20{,}000$ (away from origin)

$0 \ge 10{,}000$ (away from origin)



Table

Points	$Z = \frac{10}{100}x + \frac{9}{100}y$
A(20,000 10,000)	Z = Rs, 2900
B(40,000, 10,000)	Rs 4900 ← Maximize
C(25000, 25000)	Rs 4750
D(20,000, 20,000)	Rs 3800

So he has to invest Rs 40,000 in 'A' and Rs 10,000 in bond 'B' to get maximum return Rs 49,00

23. Find the equation of the plane which contains the line of intersection of the planes.

$\vec{r}\cdot(\hat{i} - 2\hat{j} + 3\hat{k}) - 4 = 0$ and $\vec{r}\cdot(-2\hat{i} + \hat{j} + \hat{k}) + 5 = 0$

and whose intercept on x-axis is equal to that of on y-axis.

Sol. Planes are $\vec{r}\cdot(\hat{i} - 2\hat{j} + 3\hat{k}) - 4 = 0$

$\vec{r}\cdot(-2\hat{i} + \hat{j} + \hat{k}) + 5 = 0$

$\Rightarrow$ $x - 2y + 3z - 4 = 0$

}Two planes

& $-2x + y + z + 5 = 0$

Any plane passing through the line of intersect

$$(x - 2y + 3z - 4) + \lambda\ (-2x + y + z + 5) = 0$$

$$x\,(1 - 2\lambda) + y\,(-2 + \lambda) + z\,(3 + \lambda) + (5\lambda - 4) = 0$$

Intercepts are equal on axes

so $\dfrac{4-5\lambda}{1-2\lambda} = \dfrac{4-5\lambda}{-2+\lambda}$

$\Rightarrow -2 + \lambda = 1 - 2\lambda$

$\Rightarrow 3\lambda = 3 \Rightarrow \lambda = 1$

required plane

$-x - y + 4z + 1 = 0$

$\therefore$ $\boxed{x + y - 4z - 1 = 0}$ Ans.

24. Prove that $y = \dfrac{4\sin\theta}{2+\cos\theta} - \theta$ is an increasing function of θ on $\left[0, \dfrac{\pi}{2}\right]$

OR

Show that semi-vertical angle of a cone of maximum volume and given slant height is $\cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$.

Sol. $y = \dfrac{4\sin\theta}{2+\cos\theta} - \theta,\ \theta \in [0, \pi/2]$

Diff. w. r. to θ, we get

$$\frac{dy}{d\theta} = 4\left\{\frac{(2+\cos\theta)\cos\theta - \sin\theta(-\sin\theta)}{(2+\cos\theta)^2}\right\} - 1$$

$$= 4\left\{\frac{2\cos\theta + 1}{(2+\cos\theta)^2}\right\} - 1$$

$$= \frac{8\cos\theta + 4 - (2+\cos\theta)^2}{(2+\cos\theta)^2}$$

$$= \frac{4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2}$$

$$= \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}$$

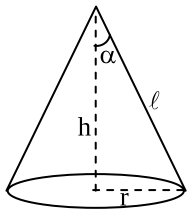
[$\because$ $0 \le \cos\theta \le 1\ \forall\ \theta \in [0, \pi/2]$]

$\Rightarrow \dfrac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2} \ge 0\ \forall\ \theta \in [0, \pi/2]$

$\Rightarrow \dfrac{dy}{d\theta} \ge 0\ \forall\ \theta \in [0, \pi/2]$

$\Rightarrow$ y is increasing in $[0, \pi/2]$

OR



Volume V = $\frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi(\ell^2 - h^2)h$

For maxima and minima

$\frac{dv}{dh} = 0$

$\frac{1}{3}\pi(\ell^2 - 3h^2) = 0$

$\Rightarrow \ell^2 = 3h^2$

$\Rightarrow \boxed{h = \frac{\ell}{\sqrt{3}}}$

and $\frac{d^2v}{dh^2} = \frac{1}{3}\pi(0 - 6h)$

$= -2\pi h$

$= -2\pi\left(\frac{\ell}{\sqrt{3}}\right) < 0$

so at $h = \frac{\ell}{\sqrt{3}}$ volume of cone is maximum

and semivertical angle α as :

$\cos\alpha = \frac{h}{\ell}$

$\cos\alpha = \frac{\ell/\sqrt{3}}{\ell}$

$\cos\alpha = \frac{1}{\sqrt{3}}$

$\therefore \boxed{\alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)}$ Ans.

25. Let A = R × R and * be a binary operation on A defined by (a, b) * (c, d) = (a + c, b + d)
Show that * is commutative and associative. Find the identity element for * on A. Also find the inverse of every element (a, b) ∈ A.

Sol. A = R × R

(a, b)*(c, d) = (a + c, b +d)

Commutative : let (a., b), (c, d) ∈A

(a, b)*(c, d) = (a + c, b + d)

= (c + a, d + b)

= (c, d) (a, b) ∀ (a, b) (c, d) ∈A

* is commutative

Associative : Let (a, b), (c, d), (e, f) ∈ A

((a, b) * (c, d)) * (e, f) = ((a + c, b + d)) * (e, f)

= (a + c + e, b + d + f)

= (a + (c + e), b + (d + f))

= (a, b) * (c + e, d + f)

= (a, b) * ((c, d) * (e, f))

∀ (a, b), (c, d) (e, f) ∈ A

* is associative

Identity element:

Let (e_1, e_2) ∈ A

is identify element for * operation by definition

⇒ (a, b) * (e_1, e_2) = (a, b)

⇒ $(a + e_1, b + e_2)$ = (a, b)

$a + e_1 = a, b + e_2 = b$

⇒ $e_1 = 0, e_2 = 0$

⇒ (0, 0) ∈A

⇒ (0, 0) is identify element for *

Inverse : let (b_1, b_2) ∈A is inverse of element (a, b)∈A then by definition.

(a, b) * (b_1, b_2) = (0, 0)

$(a + b_1, b + b_2)$ = (0, 0)

⇒ $a + b_1 = 0, b + b_2 = 0$

⇒ (– a, –b) ∈A is inverse of every element (a, b) ∈A.

26. Three numbers are selected at random (without replacement) from first six positive integers. Let X denote the largest of the three numbers obtained. Find the probability distribution of X. Also, find the mean and variance of the distribution.

Sol. First six positive integrs are 1, 2, 3, 4, 5, 6

$\therefore$ Three numbers are selected at random

without replacement so, total no. of ways ${}^6C_3 = 20$

let, x denote the larger of three numbers

so x can take values 3, 4, 5, 6

$p(x = 3) = \frac{1}{20}$

$p(x = 4) = \frac{3}{20}$

$p(x = 5) = \frac{6}{20}$

$p(x = 6) = \frac{10}{20}$

x	3	4	5	6
p(x)	$\frac{1}{20}$	$\frac{3}{20}$	$\frac{6}{20}$	$\frac{10}{20}$
xp(x)	$\frac{3}{20}$	$\frac{12}{20}$	$\frac{30}{20}$	$\frac{60}{20}$
$x^2p(x)$	$\frac{9}{20}$	$\frac{48}{20}$	$\frac{150}{20}$	$\frac{360}{20}$

$\Rightarrow$ mean $= \sum x\ p(x)$

$= \frac{3}{20} + \frac{12}{20} + \frac{30}{20} + \frac{60}{20}$

$= \frac{105}{20} = 5.25$

Varience $= \sum x^2 p(x) - \left(\sum x\, p(x)\right)^2$

$= \left(\frac{567}{20}\right) - \left(\frac{105}{20}\right)^2 = 28.35 - 27.56 = 0.79$