CAREER POINT

JEE Advance Exam 2015 (Solution)

PART I - PHYSICS

Date : 24 / 05 / 2015

SECTION – 1 (Maximum Marks : 32)

- This Section contains EIGHT questions
- The answer to each questions is a **SINGLE DIGIT INTEGER** ranging from 0 to 9, both inclusive
- For each questions, darken the bubble corresponding to the correct integer in the ORS
- Marking scheme :

+ 4 If the bubble corresponding to the answer is darkened

0 In all other cases

Q.1 A bullet is fired vertically upwards with velocity v from the surface of a spherical planet. When it reaches its maximum height, Its acceleration due to the planet's gravity is $1/4^{th}$ of its value at the surface of the planet. If the escape velocity from the planet is $v_{esc} = v\sqrt{N}$, then the value of N is (ignore energy loss due to atmosphere)

Ans. [2]

Sol. Applying conservation of mechanical energy

$$\frac{1}{2}mv^2 + \frac{-GMm}{R} = 0 - \frac{GMm}{R+h}$$

$$\frac{1}{2}mv^2 = GMm\left[\frac{1}{R} - \frac{1}{R+h}\right]$$

$$\frac{1}{2}v^2 = GM\left[\frac{h}{R\times(R+h)}\right] \qquad \text{.......(i)}$$

We have

$$g_h = g\left(1+\frac{h}{R}\right)^{-2}$$

$$\frac{g}{4} = g\left(1+\frac{h}{R}\right)^{-2}$$

$$g\,(2)^{-2} = g\left(1+\frac{h}{R}\right)^{-2}$$

$$h = R \qquad \text{.......(ii)}$$

From equation (i) & (ii)

$$\frac{1}{2}v^2 = GM\left[\frac{h}{2R^2}\right] = \frac{GMR}{2R^2} = \frac{GM}{2R}$$

$$v^2 = \frac{GM}{R}$$

$$v = \sqrt{\frac{GM}{R}}$$

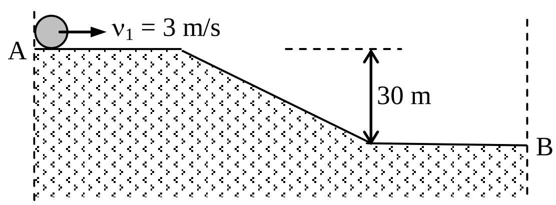
Escape velocity at surface of planet

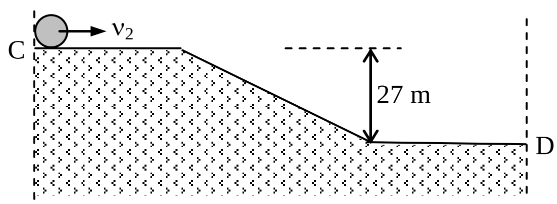
$$v_{esc} = \sqrt{\frac{2GM}{R}}$$

$v_{sec} = \sqrt{2}v$ [Given $v_{sec} = v\sqrt{N}$]

N = 2

Q.2 Two identical uniform discs roll without slipping on two different surfaces AB and CD (see figures) starting at A and C with linear speeds v_1 and v_2, respectively, and always remain in contact with the surfaces. If they reach B and D with the same linear speed and v_1 = 3m/s, then v_2 in m/s is (g = 10 m/s^2)





Ans. **[7]**

Sol. For 1st body (Applying conservation of energy)

$$\frac{1}{2}mv_1^2\left(1+\frac{k^2}{R^2}\right) = mg\,(30) + \frac{1}{2}mv^2\left(1+\frac{k^2}{R^2}\right) \quad \ldots\ldots(i)$$

For 2nd body (Applying conservation of energy)

$$\frac{1}{2}mv_2^2\left(1+\frac{k^2}{R^2}\right) = mg\,(27) + \frac{1}{2}mv^2\left(1+\frac{k^2}{R^2}\right) \quad \ldots\ldots(ii)$$

equation (i) & (ii)

$$\frac{1}{2}m\left(v_1^2 - v_2^2\right)\left(1+\frac{k^2}{R^2}\right) = mg\,(3)$$

$$\frac{1}{2}\left(9 - v_2^2\right)\left(1+\frac{1}{2}\right) = 30$$

$$\left(9 - v_2^2\right) \times \frac{3}{4} = 30$$

$$9 - v_2^2 = 40$$

$$v_2 = 7 \text{ m/s}$$

Q.3 Two spherical stars A and B emit blackbody radiation. The radius of A is 400 times that of B and A emits 10^4 times the power emitted from B. The ratio $\left(\frac{\lambda_A}{\lambda_B}\right)$ of their wavelength λ_A and λ_B at which the peaks occur in their respective radiation curves is

Ans. **[2]**

Sol. Emissive power

$P = \sigma A T^4$

$$\frac{P_1}{P_2} = \frac{\sigma A_1 T_1^4}{\sigma A_2 T_2^4} \qquad (A = 4\pi R^2)$$

$$\frac{10^4}{1} = 400 \times 400 \times \left(\frac{\lambda_B}{\lambda_A}\right)^4 \qquad \text{[using Weins displacement law } \lambda_{max} \propto \frac{1}{T}\text{]}$$

$$\frac{10^4}{(20)^4} = \left(\frac{\lambda_B}{\lambda_A}\right)^4 \Rightarrow \frac{\lambda_B}{\lambda_A} = \frac{10}{20}$$

$$\Rightarrow \frac{\lambda_B}{\lambda_A} = \frac{1}{2}$$

$$\Rightarrow \frac{\lambda_A}{\lambda_B} = \frac{2}{1}$$

Q.4 A nuclear power plant supplying electrical power to a village uses a radioactive material of half life T years as the fuel. The amount of fuel at the beginning is such that the total power requirement of the village is 12.5 % of the electrical power available from the plant at that time. If the plant is able to meet the total power needs of the village for a maximum period of n T years, then the value of n is

Ans. **[3]**

Sol. According to radioactivity $N = \frac{N_0}{2^n}$ (where $n = \frac{t}{T_H}$)

Requirement of power is 12.5% of total power. Then up to the time when radioactive nuclei are 12.5% of initial number, plant will provide the energy requirement of village

$N = 12.5\ \%\ \text{of}\ N_0$

$N = \left(\frac{12.5}{100}\right) N_0$

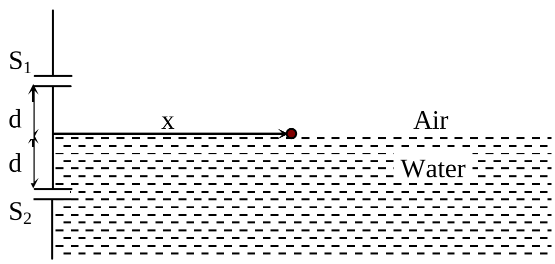
$\frac{N_0}{2^n} = \frac{N_0}{8} \Rightarrow 2^n = 8$

$\Rightarrow n = 3 \Rightarrow \frac{t}{T_H} = 3$

t = 3T

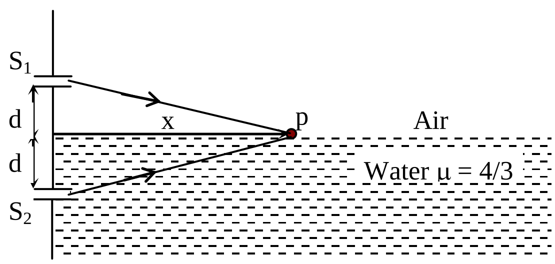
n = 3

Q.5 A Young's double slit interference arrangement with slits S_1 and S_2 is immersed in water (refractive index = 4/3) as shown in figure. The positions of maxima on the surface of water are given by $x^2 = p^2m^2\lambda^2 - d^2$, where λ is the wavelength of light in air (refractive index = 1), 2d is the separation between the slits and m is an integer. The value of p is



Ans. **[3]**

Sol.



For maxima at point p

Path difference between waves $\Delta x = m\lambda$

$$S_2p - S_1p = m\lambda$$

$$\mu\sqrt{x^2 + d^2} - \sqrt{x^2 + d^2} = m\lambda$$

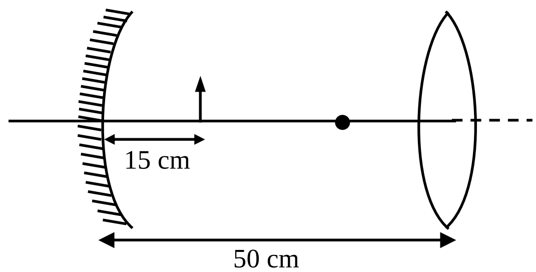
$$\sqrt{x^2 + d^2}\,(\mu - 1) = m\lambda$$

$$\sqrt{x^2 + d^2} = 3m\lambda \quad (\because \mu = \frac{4}{3})$$

$$x^2 = 3^2m^2\lambda^2 - d^2$$

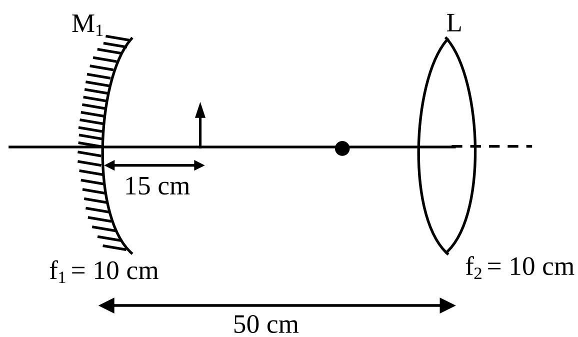
On comparing we get p = 3

Q.6 Consider a concave mirror and a convex lens (refractive index = 1. 5) of focal length 10 cm each , separated by a distance of 50 cm in air (refractive index = 1) as shown in the figure. An object is placed at a distance of 15 cm from the mirror. Its erect image formed by this combination has magnification M_1. When the set-up is kept in a medium of refractive index 7/6, the magnification becomes M_2. The magnitude $\left|\frac{M_2}{M_1}\right|$ is



Ans. **[7]**

Sol. In vacuum



For mirror u = – 15, f = – 10

then $\frac{1}{v} + \frac{1}{-15} = \frac{1}{-10}$ (According to mirror formula)

$v = -30$

$m_1 = -\left(\frac{-30}{-15}\right) = -2$

ie 1st image is formed at 30 cm before mirror. Now it will behave as object for lens.

hence for lens u = – 20 , f = 10

$\frac{1}{v} - \frac{1}{-20} = \frac{1}{10}$

$v = 20$

$m_2 = \left(\frac{20}{-20}\right) = -1$

overall magnification $M_1 = m_1 m_2 = 2$

In vacuum

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\frac{1}{10} = \left(\frac{3}{2} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\frac{1}{R_1} - \frac{1}{R_2} = \frac{1}{5}$$

In medium for lens- $\left(\mu = \frac{7}{6}\right)$

$$\frac{1}{f'} = \left(\frac{\frac{3}{2}}{\frac{7}{6}} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\frac{1}{f'} = \frac{2}{7} \times \frac{1}{5}$$

$$f' = \frac{35}{2}$$

there is no change for mirror. hence for mirror

$u = -15, f = -10$

then $v = -30$

$m_1 = -2$

For Lens

$$u = -20, f = \frac{35}{2}$$

$$\frac{1}{v} - \frac{1}{-20} = \frac{2}{35}$$

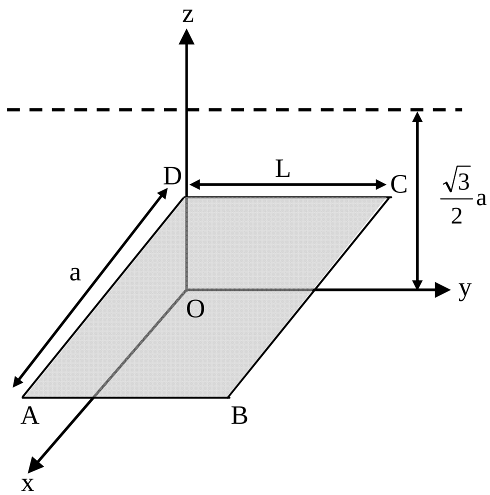
$v = 140$

$$m_2 = \left(\frac{140}{-20}\right) = -7$$

Overall magnification $M_2 = m_1 m_2 = 14$

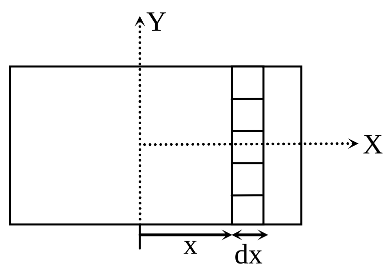
Now $\left(\frac{M_2}{M_1}\right) = \frac{14}{2} = 7$

Q.7 An infinitely long uniform line charge distribution of charge per unit length λ lies parallel to the y-axis in the y-z plane at $z = \frac{\sqrt{3}}{2}a$ (see figure). If the magnitude of the flux of the electric field through the rectangular surface ABCD lying in the x-y plane with its centre at the origin is $\frac{\lambda L}{n\varepsilon_0}$ (ε_0 = permittivity of free space), then the value of n is



Ans. **[6]**

Sol.



Electric flux though elemental part -

$$d\phi = EdA\cos\theta$$

$$d\phi = \frac{\lambda}{2\pi\epsilon_0\sqrt{x^2+\frac{3a^2}{4}}} \times Ldx \times \frac{\frac{\sqrt{3}a}{2}}{\sqrt{x^2+\frac{3a^2}{4}}}$$

$$\int d\phi = \frac{\sqrt{3}a\lambda L}{4\pi\epsilon_0}\int_{x=-\frac{a}{2}}^{a/2}\frac{dx}{\left(x^2+\frac{3a^2}{4}\right)}$$

$$\phi = \frac{\sqrt{3}a\lambda L}{4\pi\epsilon_0} \times 2 \int_0^{a/2} \frac{dx}{x^2 + \frac{3a^2}{4}}$$

$$\phi = \frac{\sqrt{3}a\lambda L}{2\pi\epsilon_0} \times \frac{2}{\sqrt{3a}}\left[\tan^{-1}\left(\frac{2x}{\sqrt{3a}}\right)\right]_0^{a/2}$$

$$\phi = \frac{\lambda L}{\pi\epsilon_0}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\phi = \frac{\lambda L}{6\epsilon_0}$$

On comparing n = 6

Q.8 Consider a hydrogen atom with its electron in the n^{th} orbital. An electromagnetic radiation of wavelength 90 nm is used to ionize the atom. If the kinetic energy of the ejected electron is 10.4 eV, then value of n is (hc = 1242 eV nm)

Ans. **[2]**

Sol. Energy of photon $E_{Ph} = \frac{1242}{\lambda(\text{in nm})}$ eV

$\lambda = 90$ nm

$E_{Ph} = \frac{1242}{90}$ eV $\Rightarrow E_{Ph} = 13.8$ eV

K.E. of ejected electron K.E. = 10.4 eV

So let energy of the orbits is E_n

$E_n + E_{Ph} = 10.4$ eV

$E_n + 13.8 = 10.4$

$\boxed{E_n = -3.4 \text{ eV}}$

$E_n = -13.6 \frac{Z^2}{n^2}$ eV

for H-atom (Z = 1)

$-3.4 = -13.6 \frac{(1)^2}{n^2}$

n = 2

SECTION – 2 (Maximum Marks : 40)

- This Section contains **TEN** questions
- Each question has **FOUR** options (A), (B), (C) and (D) **ONE OR MORE THAN ONE** of these four option(s) is (are) correct
- For each questions, darken the bubble(s) corresponding to all the correct option (s) in the **ORS**
- Marking scheme :

+4 If only the bubble (s) corresponding to all the correct option (s) is (are) darkened

0 If none of the bubbles is darkened

–2 In all other cases

Q.9 A container of fixed volume has a mixture of one mole of hydrogen and one mole of helium in equilibrium at temperature T. Assuming the gases are ideal, the correct statement (s) is (are)

(A) The average energy per mole of the gas mixture is 2RT.

(B) The ratio of speed of sound in the gas mixture to that in helium gas is $\sqrt{6/5}$

(C) The ratio of the rms speed of helium atoms to that of hydrogen molecules is 1/2.

(D) The ratio of the rms speed of helium atoms to that of hydrogen molecules is $1/\sqrt{2}$.

Ans. **[A,B,D]**

Sol. (A) $(KE_{av})_{mix}/_{mol} = \dfrac{\frac{5}{1}(1)\,RT + \frac{3}{2}(1)RT}{2} = 2RT$

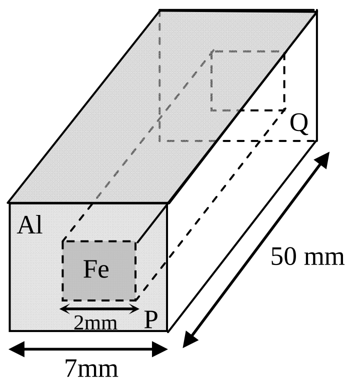
(B) $v_{sound} = \sqrt{\dfrac{\gamma RT}{M}}$

Here, $\gamma_{mix} = \dfrac{\left(1\times\frac{5}{2}\right)+\left(1\times\frac{7}{2}\right)}{\left(1\times\frac{3}{2}\right)+\left(1\times\frac{5}{2}\right)} = \dfrac{3}{2}$

$$\Rightarrow \frac{v_{mix}}{v_{He}} = \sqrt{\frac{\gamma_{mix}}{\gamma_{He}}}\sqrt{\frac{M_{He}}{M_{mix}}}$$

$$\frac{v_{mix}}{v_{He}} = \sqrt{\frac{\left(\frac{3}{2}\right)}{\left(\frac{5}{3}\right)}} \times \sqrt{\frac{4}{3}} = \sqrt{\frac{6}{5}}$$

Q.10 In an aluminum (Al) bar of square cross section, a square hole is drilled and is filled with iron (Fe) as shown in the figure. The electrical resistivities of Al and Fe are 2.7×10^{-8} Ωm and 1.0×10^{-7} Ωm, respectively. The electrical resistance between the two faces P and Q of the composite bar is



(A) $\frac{2475}{64}\mu\Omega$

(B) $\frac{1875}{64}\mu\Omega$

(C) $\frac{1875}{49}\mu\Omega$

(D) $\frac{2475}{132}\mu\Omega$

Ans. **[B]**

Sol. For Aluminium

$$R_{Al} = \frac{\rho_{Al}\ell}{A_{Al}} = \frac{2.7\times10^{-8}\times50\times10^{-3}}{(49-4)\times10^{-6}}$$

$R_{Al} = 30 \times 10^{-6}\ \Omega$

For iron Fe

$R_{Fe} = \frac{\rho_{Fe}\ell}{A_{Fe}}$

$R_{Fe} = \frac{10^{-7} \times 50 \times 10^{-3}}{4 \times 10^{-6}}$

$R_{Fe} = 1250 \times 10^{-6}\,\Omega$

$Req. = \frac{R_{A1}R_{Fe}}{R_{A1} + R_{Fe}}$

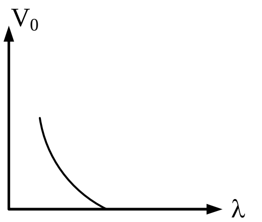
$Req. = \frac{30 \times 10^{-6} \times 1250 \times 10^{-6}}{(30 + 1250) \times 10^{-6}}$

$Req. = \frac{30 \times 1250}{1280} \times 10^{-6}$

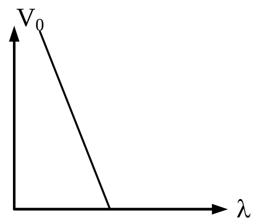
$Req. = \frac{1875}{64}\mu\Omega$

Q.11 For photo-electric effect with incident photon wavelength λ, the stopping potential is V_0. Identify the correct variation(s) of V_0 with λ and $1 / \lambda$.

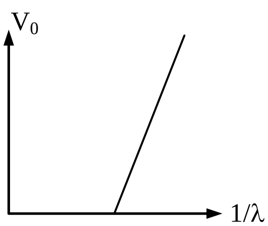
(A)



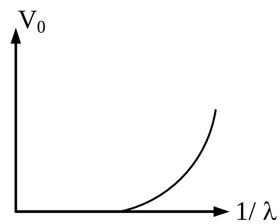
(B)



(C)



(D)



Ans. **[A,C]**

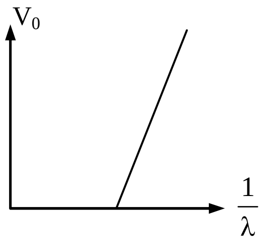
Sol. Einstein equation of photoelectric effect

$\frac{hc}{\lambda} - \phi = KE_{max}$

$\frac{hc}{\lambda} - \phi = eV_0$

$V_0 = \frac{hc}{e} \cdot \frac{1}{\lambda} - \frac{\phi}{e}$

Straight line graph between V_0 and $\frac{1}{\lambda}$



Ans. (C)

As λ increase V_0 decreases

Check for the slope of V_0 v/s λ graph

$$\text{Slope} = \frac{dV_0}{d\lambda} = \frac{-hc}{e\lambda^2}$$

as λ increase slope decreases and it is always negative

∴ Option (A) is correct

Q.12 Consider a Vernier calipers in which each 1 cm on the main scale is divided into 8 equal divisions and a screw gauge with 100 divisions on its circular scale. In the Vernier calipers, 5 divisions of the Vernier scale coincide with 4 divisions on the main scale and in the screw gauge, one complete rotation of the circular scale moves it by two divisions on the linear scale. Then :

(A) If the pitch of the screw gauge is twice the least count of the Vernier calipers, the least count of the screw gauge is 0.01 mm.

(B) If the pitch of the screw gauge is twice the least count of the Vernier calipers, the least count of the screw gauge is 0.005 mm.

(C) If the least count of linear scale of the screw gauge is twice the least count of the Vernier calipers, the least count of the screw gauge is 0.01 mm.

(D) If the least count of the linear scale of the screw gauge is twice the least count of the Vernier calipers, the least count of the screw gauge is 0.005 mm.

Ans. **[B,C]**

Sol. Vernier scale least count calculations

$$\text{M.S.D.} = \text{Main scale division} = \frac{1}{8}\ \text{cm}$$

V.S.D. = Vernier scale division

L.C. = M.S.D. – V.S.D.

5 × V.S.D. = 4 × M.S.D.

$$\text{V.S.D.} = \frac{4}{5}\ \text{M.S.D.} = \frac{4}{5} \times \frac{1}{8} = \frac{1}{10}\ \text{cm}$$

L.C. = $\frac{1}{8} - \frac{1}{10}$ $\Rightarrow \frac{2}{80}$ $\Rightarrow \frac{1}{40}$ cm $\Rightarrow$ 0.25 cm

If pitch of screw gauge = 2 × 0.25 = 0.5 mm

then L.C. of screw gauge = $\frac{0.5}{100}$ = 0.005 mm

option B is ans.

If L.C. of linear scale of screw gauge = 2 × 0.25 $\Rightarrow$ 0.5 mm

then L.C. of screw gauge = $\frac{2 \times 0.5}{100}$ = 0.01 mm

Q.13 Planck's constant h, speed of light c and gravitational constant G are used to form a unit of length L and a unit of mass M. Then the correct option(s) is(are)

(A) $M \propto \sqrt{c}$

(B) $M \propto \sqrt{G}$

(C) $L \propto \sqrt{h}$

(D) $L \propto \sqrt{G}$

Ans. **[A,C,D]**

Sol. $E = \frac{hc}{\lambda}$

$E = \frac{-Gm_1m_2}{r}$

$\frac{hc}{\lambda} = \frac{GM^2}{r}$

unit of λ and r is L

So $M = \left(\frac{hc}{G}\right)^{1/2}$ …(1)

$M \propto \sqrt{c}$ $M \propto \frac{1}{\sqrt{G}}$ Ans. (A)

$E = mc^2$

$E = \frac{hc}{\lambda}$

$\frac{hc}{\lambda} = mc^2$

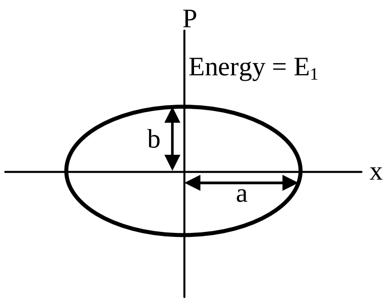
$\lambda = \dfrac{h}{m}$

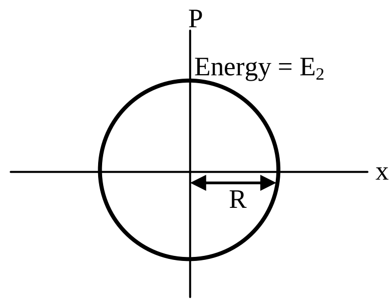
$\therefore\ L = \dfrac{h}{M}$

$L = \dfrac{h}{(hc)^{1/2}} G^{1/2}$ from (1)

$\therefore\ L \propto \sqrt{G}$ $L \propto \sqrt{h}$

Q.14 Two independent harmonic oscillators of equal mass are oscillating about the origin with angular frequencies ω_1 and ω_2 and have total energies E_1 and E_2, respectively. The variations of their momenta p with positions x are shown in the figures. If $\dfrac{a}{b} = n^2$ and $\dfrac{a}{R} = n$, then the correct equation(s) is(are)





(A) $E_1\omega_1 = E_2\omega_2$

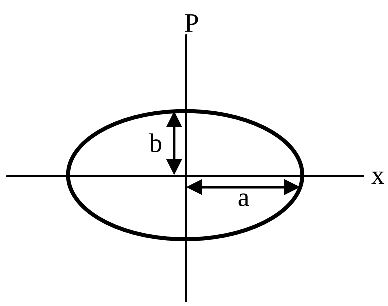
(B) $\dfrac{\omega_2}{\omega_1} = n^2$

(C) $\omega_1\omega_2 = n^2$

(D) $\dfrac{E_1}{\omega_1} = \dfrac{E_2}{\omega_2}$

Ans. **[B,D]**

Sol. I^{st} harmonic oscillator.



$$\frac{P^2}{b^2} + \frac{x^2}{a^2} = 1$$

$$P^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$P = \frac{b}{a}\sqrt{a^2 - x^2}$$

$\Rightarrow\ v = \dfrac{P}{m}$

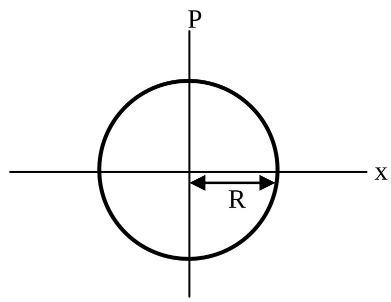
$v = \dfrac{b}{am}\sqrt{a^2 - x^2}$

Comparing $\quad v = \omega\sqrt{A^2 - x^2}$

$\omega_1 = \dfrac{b}{am}$, $A_1 = a$

& $E_1 = \dfrac{1}{2}\, m\, \omega_1^2\ A_1^2$

II[nd] harmonic oscillation



$P^2 + x^2 = R^2$

$P = \sqrt{R^2 - x^2}$

$v = \dfrac{P}{m}$

$v = \dfrac{1}{m}\sqrt{R^2 - x^2} \qquad (v = \omega\sqrt{A^2 - x^2})$

Comparing $\quad \omega_2 = \dfrac{1}{m}$, $A_2 = R_1$

$E_2 = \dfrac{1}{2}\, m\, \omega_2^2\ A_2^2$

(1) $\quad \dfrac{\omega_2}{\omega_1} = \dfrac{\frac{1}{m}}{\frac{b}{am}}$

$\Rightarrow \quad \dfrac{\omega_2}{\omega_1} = \dfrac{a}{b} \qquad$ (given $\dfrac{a}{b} = n^2$)

$\dfrac{\omega_2}{\omega_1} = n^2$

(2) $\quad \dfrac{E_1}{\omega_1} = \dfrac{1}{2}\, m\omega_1\ A_1^2$

$\Rightarrow \quad \frac{E_1}{\omega_1} = \frac{1}{2} m \left(\frac{b}{am}\right) (a)^2$

$\frac{E_1}{\omega_1} = \frac{1}{2} ab$

& $\quad \frac{E_2}{\omega_2} = \frac{1}{2} m (\omega_2) A_2^2$

$\frac{E_2}{\omega_2} = \frac{1}{2} m \left(\frac{1}{m}\right) (R)^2$

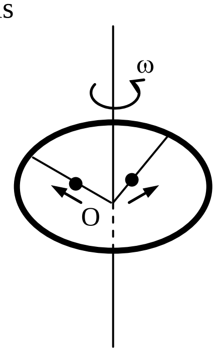
$\frac{E_2}{\omega_2} = \frac{R^2}{2}$

the value of ab = $\frac{a^2}{n^2} \quad \left(\frac{a}{b} = n^2\right)$

& $\quad R^2 = \frac{a}{n^2} \quad \left(\frac{a}{R} = n\right)$

So $\quad \frac{E_1}{\omega_1} = \frac{E_2}{\omega_2}$

Q.15 A ring of mass M and radius R is rotating with angular speed ω about a fixed vertical axis passing through its centre O with two point masses each of mass $\frac{M}{8}$ at rest at O. These masses can move radially outwards along two massless rods fixed on the ring as shown in the figure. At some instant the angular speed of the system is $\frac{8}{9}\omega$ and one of the masses is at a distance of $\frac{3}{5}$R from O. At this instant the distance of the other mass from O is



(A) $\frac{2}{3}R$ (B) $\frac{1}{3}R$ (C) $\frac{3}{5}R$ (D) $\frac{4}{5}R$

Ans. **[C or D]**

Sol. This question is based on angular momentum conservation

As given in the question initial angular velocity of ring is ω and final angular velocity of system is $\frac{8}{9}\omega$.

so,

$L_i = L_f$

$I\omega = L_f$

$I_{ring}\ \omega = L_f$

$MR^2\ \omega = MR^2 \times \frac{8}{9}\omega + \frac{M}{8} \times \left(\frac{3}{5}R\right)^2 \times \frac{8}{9}\omega + \frac{M}{8} x^2 \times \frac{8}{9}\omega$

$$R^2 \times 1 = \frac{8R^2}{9} + \frac{1}{8} \times \frac{9}{25} \times \frac{8R^2}{9} + \frac{x^2}{9}$$

$$R^2 \left[1 - \frac{8}{9} - \frac{1}{25}\right] = \frac{x^2}{9}$$

$$R^2 \left[\frac{25 \times 9 - 8 \times 25 - 9}{9 \times 25}\right] = \frac{x^2}{9}$$

$$R^2 \frac{16}{25} = x^2$$

$$x = \frac{4}{5} R$$

So, option (D) is correct

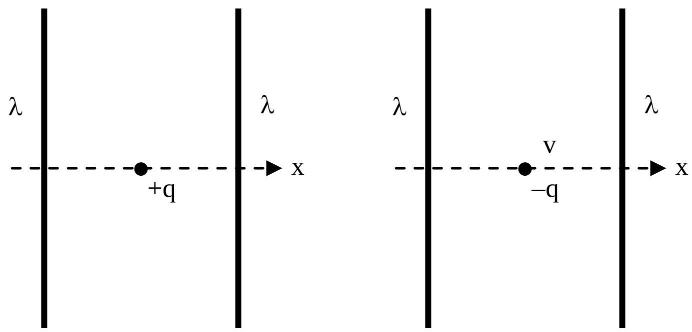
OR

Now we assume ring start to rotate at t = 0 by certain external agent.

Because both rods are frictionless and mass of the particles are equal. At t = 0, both particle starts to move from centre so they will experience same force at all the time. Hence position of particle will be same. If first particle is at a distance $\frac{3}{5}R$ from centre then other will be also at a distance $\frac{3}{5}R$.

So, option (C) is also correct.

Q.16 The figures below depict two situations in which two infinitely long static line charges of constant positive line charge density λ are kept parallel to each other. In their resulting electric field, point charges q and –q are kept in equilibrium between them. The point charges are confined to move in the x direction only. If they are given a small displacement about their equilibrium positions, then the correct statements(s) is(are)



(A) Both charges execute simple harmonic motion.

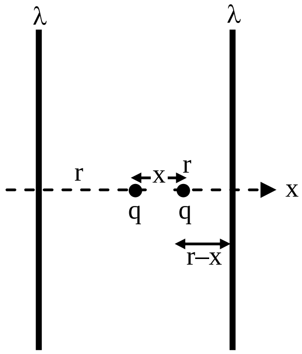
(B) Both charges will continue moving in the direction of their displacement.

(C) Charge +q executes simple harmonic motion while charge –q continues moving in the direction of its displacement.

(D) Charge –q executes simple harmonic motion while charge +q continues moving in the direction of its displacement.

Ans. **[C]**

Sol.



If q is displaced slightly along x axis it will come back to original position

F_{net} on q on displacing by small displacement x = restoring force = $\dfrac{\lambda q}{2\pi\varepsilon_0(r-x)} - \dfrac{\lambda q}{2\pi\varepsilon_0(r+x)}$

$$\Rightarrow F = \frac{q\lambda}{2\pi\varepsilon_0}\left[\frac{1}{r-x} - \frac{1}{r+x}\right]$$

$$\Rightarrow F = \frac{q\lambda \times 2x}{2\pi\varepsilon_0(r^2 - x^2)} \qquad \text{neglecting x}$$

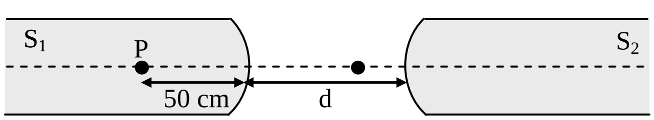
$$F_{restoring} = \frac{q\lambda x}{\pi\varepsilon_0 r^2}$$

$$a = \frac{q\lambda x}{m\pi\varepsilon_0 r^2}$$

$a \propto x$ so q will do SHM

– q will not able to oscillation as F_{net} on – q will not send it back to its original equilibrium position. – q continue moving in direction of its displacement.

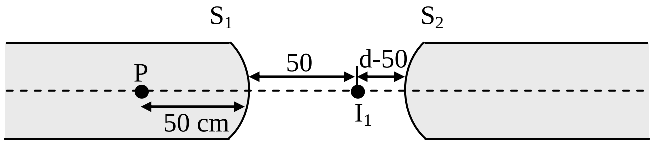
Q.17 Two identical glass rods S_1 and S_2 (refractive index = 1.5) have one convex end of radius of curvature 10 cm. They are placed with the curved surfaces at a distance d as shown in the figure, with their axes (shown by the dashed line) aligned. When a point source of light P is placed inside rod S_1 on its axis at a distance of 50 cm from the curved face, the light rays emanating from it are found to be parallel to the axis inside S_2. The distance d is



(A) 60 cm
(B) 70 cm
(C) 80 cm
(D) 90 cm

Ans. **[B]**

Sol.



This question is of multiple event.

First event on curved surface of S_1 & Second on curved surface of S_2

Event – 1 $n_2 = 1$

$n_1 = 1.5$

$$\frac{1}{v} - \frac{1.5}{u} = \frac{1-1.5}{R}$$

$$\frac{1}{v} - \frac{1.5}{-50} = \frac{-0.5}{-10}$$

$$\frac{1}{v} + \frac{1.5}{50} = \frac{1}{20}$$

$$\frac{1}{v} = \frac{1}{20} - \frac{1.5}{50}$$

$$\frac{1}{v} = \frac{5-3}{100} = \frac{2}{100} \qquad \therefore\ v = 50 \text{ cm}$$

for second event $u = -(d-50)$

$v = \infty$

$R = +10$ cm

$$\frac{1.5}{v} - \frac{1}{u} = \frac{1.5-1}{R}$$

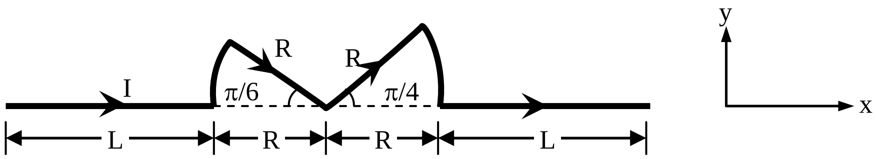
$$\frac{1.5}{\infty} - \frac{1}{-(d-50)} = \frac{0.5}{10}$$

$$\frac{1}{d-50} = \frac{1}{20}$$

$20 = d - 50$

$d = 70$ cm

Q.18 A conductor (shown in the figure) carrying constant current I is kept in the x-y plane in a uniform magnetic field $\vec{B}$. If F is the magnitude of the total magnetic force acting on the conductor, then the correct statement(s) is(are)



(A) If B is along $\hat{z}$, $F \propto (L + R)$

(B) If $\vec{B}$ is along $\hat{x}$, $F = 0$

(C) If $\vec{B}$ is along $\hat{y}$, $F \propto (L + R)$

(D) If $\vec{B}$ is along $\hat{z}$, $F = 0$

Ans. **[A,B,C]**

Sol. Option A if B is along z axis i.e $B\hat{k}$

$\vec{i\ell} = 2i\,(L + R)\,\hat{i}$

$\vec{F} = \vec{i\ell} \times \vec{B}$ $\Rightarrow$ $F = 2i\,(L + R)\,\hat{i} \times \vec{B}\,\hat{k}$

$\therefore$ $F \propto (L + R)$ option A is correct answer.

Option B if B is along x axis i.e $B\hat{i}$

then $F = 2i\,(L + R)\,\hat{i} \times (B\hat{i}) = 0$ option B is correct answer

Option C if B is along y axis i.e. $B\hat{j}$

then $F = 2i\,(L + R)\,\hat{i} \times B\hat{j}$

$|F| \Rightarrow 2i\,(L + R)\,B$

$\therefore$ $F \propto (L + R)$ option (C) is correct answer.

SECTION – 3 (Maximum Marks 16)

- This section contains **TWO** questions
- Each question contains two columns, **Column I** and **Column II**
- **Column I** has **four** entries (A), (B), (C) and (D)
- **Column II** has **five** entries (P), (Q), (R), (S) and (T)
- Match the entries in **Column I** with the entries in **Column II**
- One or more entries in **Column I** may match with one or more entries in **Column II**
- The ORS contains a 4 × 5 matrix whose layout will be similar to the one shown below:

(A)	(P)	(Q)	(R)	(S)	(T)
(B)	(P)	(Q)	(R)	(S)	(T)
(C)	(P)	(Q)	(R)	(S)	(T)
(D)	(P)	(Q)	(R)	(S)	(T)

- For each entry in **Column I**, darken the bubbles of all the matching entries. For example, if entry (A) in **Column I** matches with entries (Q), (R) and (T), then darken these three bubbles in the ORS. Similarly, for entries (B), (C) and (D).
- **Marking scheme:**

<u>For each entry in **Column I.**</u>

+2 If only the bubble(s) corresponding to all the correct match(es) is(are) darkened

0 If none of the bubbles is darkened

–1 In all other cases

Q.19 Match the nuclear processes given in column I with the appropriate option(s) in column II.

Column I	**Column II**
(A) Nuclear fusion	(P) Absorption of thermal neutrons by $^{235}_{92}U$
(B) Fission in a nuclear reactor	(Q) $^{60}_{27}Co$ nucleus
(C) β-decay	(R) Energy production in stars via hydrogen conversion to helium
(D) γ-ray emission	(S) Heavy water
	(T) Neutrino emission

Ans. **[A→R ; B → P,S ; C→ T ; D → P,Q,R,T]**

Sol. Above matching is based on theoretical concept of nuclear reaction and radioactivity.

Q.20 A particle of unit mass is moving along the x-axis under the influence of a force and its total energy is conserved. Four possible forms of the potential energy of the particle are given in column I (a and U_0 are constants). Match the potential energies in column I to the corresponding statements(s) in column II.

Column I	Column II
(A) $U_1(x) = \frac{U_0}{2}\left[1-\left(\frac{x}{a}\right)^2\right]^2$	(P) The force acting on the particle is zero at x = a.
(B) $U_2(x) = \frac{U_0}{2}\left(\frac{x}{a}\right)^2$	(Q) The force acting on the particle is zero at x = 0.
(C) $U_3(x) = \frac{U_0}{2}\left(\frac{x}{a}\right)^2 \exp\left[-\left(\frac{x}{a}\right)^2\right]$	(R) The force acting on the particle is zero at x = –a.
(D) $U_4(x) = \frac{U_0}{2}\left[\frac{x}{a}-\frac{1}{3}\left(\frac{x}{a}\right)^3\right]$	(S) The particle experiences an attractive force towards x = 0 in the region $\lvert x\rvert < a$.
	(T) The particle with total energy $\frac{U_0}{4}$ can oscillate about the point x = –a.

Ans. **[((A → P,Q,R,T) (B →Q,S) (C → P,Q,R,S) (D → P,R,T))]**

Sol. (A) $U_1 = \frac{U_0}{2}\left[1-\left(\frac{x}{a}\right)^2\right]^2$

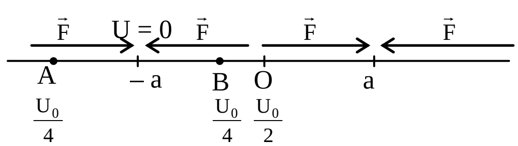
$$F = -\frac{dU_1}{dx} = \frac{-U_0}{2}\left[2\left(1-\left(\frac{x}{a}\right)^2\right)\right]\left(-\frac{2x}{a^2}\right)$$

$$F = \frac{2xU_0}{a^2}\left(1-\frac{x^2}{a^2}\right)$$

at x = a, x = 0, x = – a F = 0

if x = + a/2

F = + ve



Particle oscillating about x = – a. Shown energy in graph are potential energy and A and B are extreme point of oscillation where total energy is potential energy.

Option → P, Q, R, T correct

(B) $U_2 = \frac{U_0}{2}\left[\frac{x}{a}\right]^2$

$$F = -\frac{dU_2}{dx} = -\frac{U_0}{2a^2}(2x)$$

at x = 0 F = 0

if x = + a/2 F = – ve

towards origin

Option (Q, S) correct.

(C) $U_3 = \frac{U_0}{2}\left(\frac{x}{a}\right)^2 e^{-\left(\frac{x}{a}\right)^2}$

$$F = -\frac{dU_3}{dx} = \left(-\frac{U_0}{2a^2}\right)\frac{d}{dx}\left[x^2 e^{-\frac{x^2}{a^2}}\right]$$

$$F = -\frac{U_0}{2a^2}\left[x^2 e^{-\frac{x^2}{a^2}} - \frac{2x}{a^2} + 2x\,e^{-\frac{x^2}{a^2}}\right]$$

$$F = -\frac{U_0}{2a^2}\ e^{-\frac{x^2}{a^2}}\left[-\frac{2x^3}{a^2} + 2x\right]$$

$$F = -\frac{U_0}{2a^2}\ e^{-\frac{x^2}{a^2}}\ (+\,2x)\left[1 - \frac{x^2}{a^2}\right]$$

at $x = 0, a, -a$ $\quad F = 0$

if $x = +a/2$ $\quad F = -\text{ve}$ $\quad$ attractive

$\Rightarrow$ P, Q, R, S Option correct

(D) $U_4 = \frac{U_0}{2}\left[\frac{x}{a} - \frac{1}{3}\frac{x^3}{a^3}\right]$

$$F = -\frac{dU_4}{dx} = -\frac{U_0}{2}\left[\frac{1}{a} - \frac{1}{3a^3}(3x^2)\right]$$

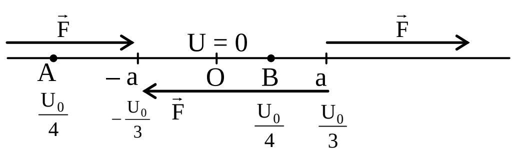
$$F = -\frac{U_0}{2}\left[\frac{1}{a} - \frac{x^2}{a^3}\right]$$

$$F = -\frac{U_0}{2a^3}\ (a^2 - x^2)$$

at $x = \pm a$ $\quad \Rightarrow F = 0$

at $\begin{matrix} x = +a/2 & F = -\text{ve} \\ x = -a/2 & F = -\text{ve} \end{matrix}\Big\}$

P, R, T correct



Oscillation will be above x = – a. For Total energy $\frac{U_0}{4}$. The extreme points will be A & B where potential energy will be maximum. (In diagram P.E. are shown).

PART II - CHEMISTRY

SECTION – 1 (Maximum Marks : 32)

- This section contains **EIGHT** questions.
- The answer to each questions is a **SINGLE DIGIT INTEGER** ranging from 0 to 9, both inclusive.
- For each question, darken the bubble corresponding to the correct integer in the **ORS**
- Marking scheme.

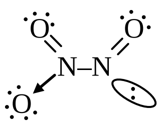
  +4 If the bubble corresponding to the answer is darkened.

  0 In all other cases.

Q.21 The total number of lone pairs of electrons in N_2O_3 is -

Ans. **[8]**

Sol. N_2O_3



Number of ℓone pairs = 8

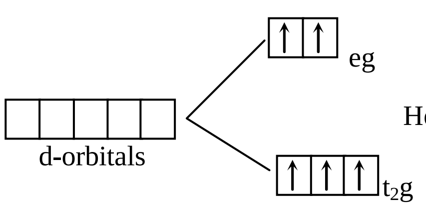
Q.22 For the octahedral complexes of Fe^{3+} in SCN^- (thiocyanato-S) and in CN^- ligand environments, the difference between the spin-only magnetic moments in Bohr Magneton (when approximated to the nearest integer) is -

[Atomic number of Fe = 26]

Ans. **[4]**

Sol. $Fe^{+3} + SCN^- \rightarrow [Fe(SCN)_6]^{3-}$

$Fe^{+3} \Rightarrow [Ar]\ 3d^5$



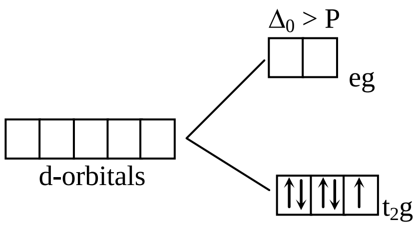
Here $\Delta_0 < P$

Number of unpaired electrons = 5

$\mu = 5.9$

$Fe^{+3} + CN^- \rightarrow [Fe(CN)_6]^{3-}$

$Fe^{+3} = [Ar]3d^5$



Number of unpaired electrons = 1

$\mu = 1.732$

Difference = 5.91 – 1.73 = 4.18

$\simeq 4$

Q.23 Among the triatomic molecules/ions, $BeCl_2$, N_3^-, N_2O, NO_2^+, O_3, SCl_2, ICl_2^-, I_3^- and XeF_2, the total number of linear molecule(s)/ion(s) where the hybridization of the central atom does not have contribution from the d-orbital(s) is.

[Atomic number : S = 16, Cl = 17, I = 53 and Xe = 54]

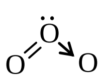
Ans. **[4]**

Sol. Cl–Be–Cl $N\equiv N:\rightarrow O$

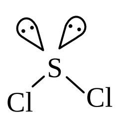
sp, linear   sp, linear

$\overset{-}{N} = \overset{+}{N} = \overset{-}{N}$ $O = \overset{+}{N} = O$

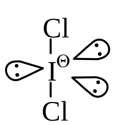
sp, linear   sp, linear



sp^2, V-shape



V-shape



sp^3d, Linear

like I_3^-, $XeF_2 \rightarrow sp^3d$, Linear

Ans. = 4

Q.24 Not considering the electronic spin, the degeneracy of the second excited state (n = 3) of H atom is 9, while the degeneracy of the second excited state of H^- is -

Ans. **[3]**

Sol. 2^{nd} excited state H^- = 2p state

$\therefore$ degeneracy of second excited state of H^- = 3

(This is not is jee advance syllabus)

Q.25 All the energy released from the reaction X → Y, $\Delta_r G^o$ = –193 kJ mol^{-} is used for oxidizing M^+ as $M^+ \rightarrow M^{3+} + 2e^-$, E^o = –0.25 V. Under standard conditions, the number of moles of M^+ oxidized when one mole of X is converted to Y is

[F = 96500 C mol^{-1}]

Ans. **[4]**

Sol. $M^+ \rightarrow M^{3+} + 2e^-$

$nM^+ \rightarrow nM^{3+} + 2ne^-$ $\quad \Delta G^o = -2n\,FE^o$

And :

$|\Delta_r G^o| = |\Delta G^o|$

$\therefore\ 193 \times 1000 = 2 \times n \times 96500 \times 0.25$

$\boxed{n = 4}$

Q.26 If the freezing point of a 0.01 molal aqueous solution of a cobalt(III) chloride-ammonia complex (which behaves as a strong electrolyte) is – 0.0558ºC, the number of chloride(s) in the coordination sphere of the complex is

[K_f of water = 1.86 K kg mol^{-1}]

Ans. **[1]**

Sol. $\Delta T_f = i.K_f.m$

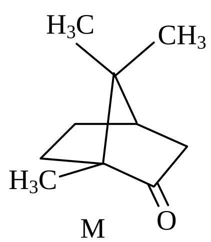
0.0558 = n × 1.86 × 0.01

n = 3

$\therefore\ [Co(NH_3)_5Cl]Cl_2$

i.e. number of chloride ion in complex is 1.

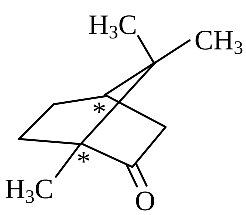
Q.27 The total number of stereoisomers that can exist for M is



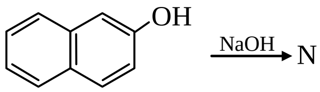
M

Ans. **[2]**

Sol. Due to it's rigid structure camphor form only two optical isomers (stereoisomer)

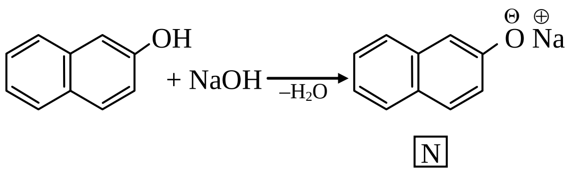


Q.28 The number of resonance structures for N is

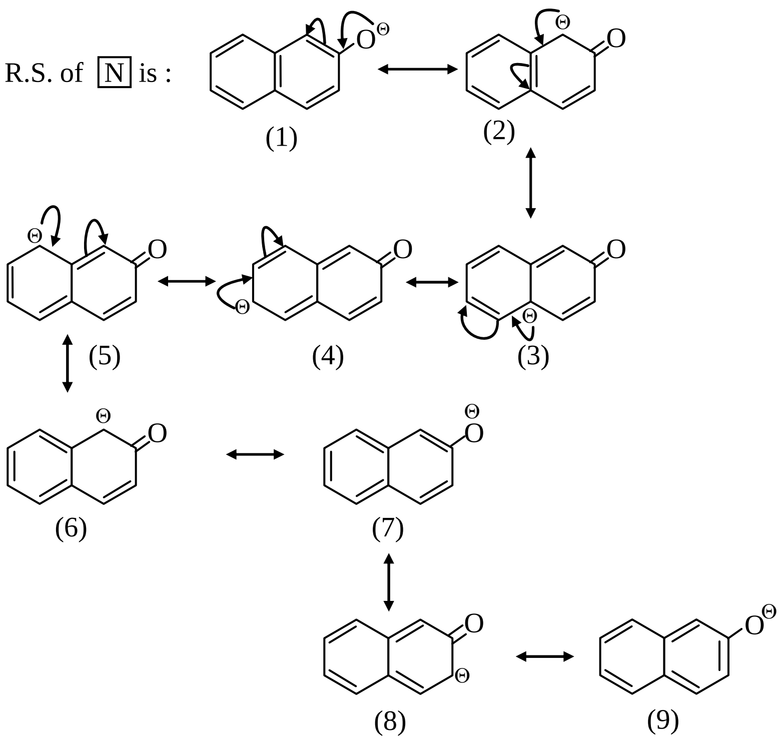


Ans. **[9]**

Sol.



R.S. of N is :



SECTION – 2 (Maximum Marks : 40)

- This section contains **TEN** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is(are) correct.
- For each question, darken the bubble(s) corresponding to all the correct option(s) in the **ORS**.
- Marking scheme.

  +4 If only the bubble(s) corresponding to all the correct option(s) is(are) darkened

  0 If none of the bubbles is darkened

  –2 In all other cases.

Q.29 Fe^{3+} is reduced to Fe^{2+} by using -

(A) H_2O_2 in presence of NaOH

(B) Na_2O_2 in water

(C) H_2O_2 in presence of H_2SO_4

(D) Na_2O_2 in presence of H_2SO_4

Ans. **[A,B]**

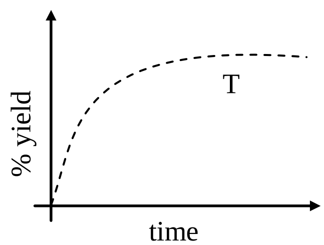
Sol. $Fe^{+3} + H_2O_2 + NaOH \rightarrow Fe^{+2} + O_2$

$$Fe^{+3} + \begin{bmatrix} Na_2O_2 + H_2O \\ \downarrow \\ NaOH + H_2O_2 \end{bmatrix} \rightarrow Fe^{+2} + O_2$$

Q.30 The %yield of ammonia as a function of time in the reaction

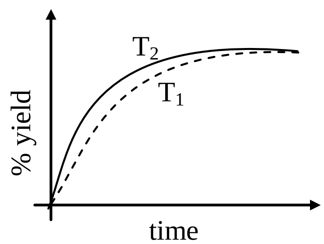
$N_2(g) + 3H_2(g) \rightleftharpoons 2NH_3(g)$, $\Delta H < 0$

at (P, T_1) is given below.

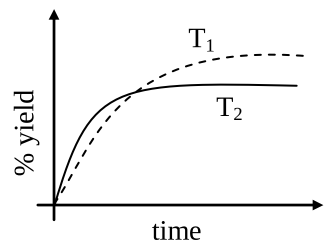


If this reaction is conducted at (P, T_2), with $T_2 > T_1$, the %yield of ammonia as a function of time is represented by -

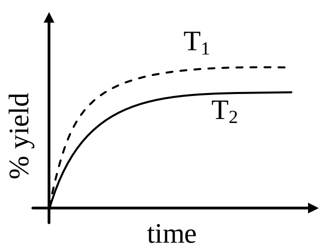
(A)



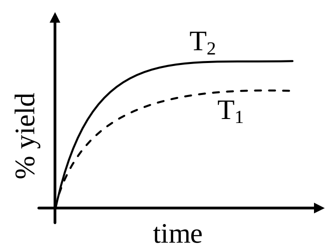
(B)



(C)



(D)



Ans. **[B]**

Sol. Initially larger is the temperature greater is the yield but due to exothermic reaction as the time lapse yield become less.

Q.31 If the unit cell of a mineral has cubic close packed (ccp) array of oxygen atoms with **m** fraction of octahedral holes occupied by aluminium ions and **n** fraction tetrahedral holes occupied by magnesium ions, **m** and **n**, respectively, are -

(A) $\frac{1}{2}, \frac{1}{8}$

(B) $1, \frac{1}{4}$

(C) $\frac{1}{2}, \frac{1}{2}$

(D) $\frac{1}{4}, \frac{1}{8}$

Ans. **[A]**

Sol.

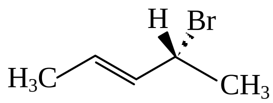
T.V.	O.V.	c.c.p.
Mg^{2+}	Al^{3+}	O^{2-}
8n	4m	4

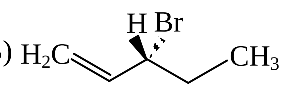
$\therefore\ 8n \times (+2) + 4m \times (+3) + 4 \times (-2) = 0$

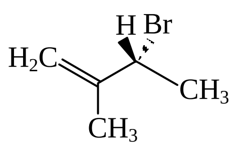
On verifying option, we get

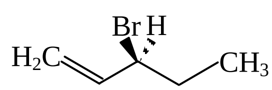
$n = \frac{1}{8}$, $m = \frac{1}{2}$

Q.32 Compound(s) that on hydrogenation produce(s) optically inactive compound(s) is / are -

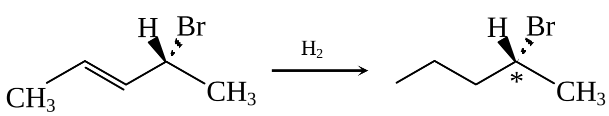
(A) 

(B) 

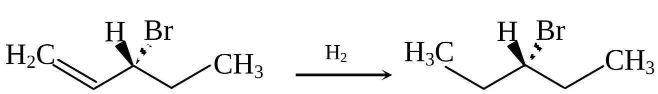
(C) 

(D) 

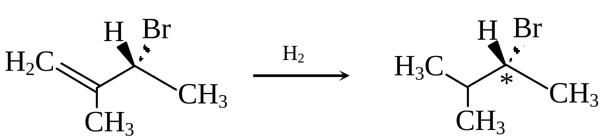
Ans. **[B,D]**

Sol. (A) 

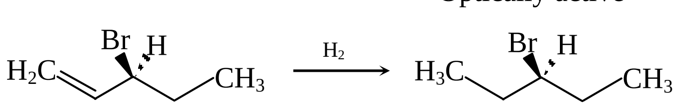
Optically active

(B) 

Optically inactive

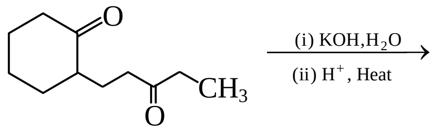
(C) 

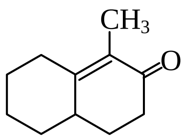
Optically active

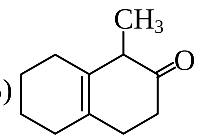
(D) 

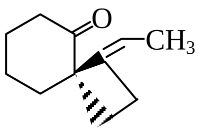
Optically inactive

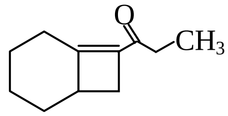
Q.33 The major product of the following reaction is -



(A) 

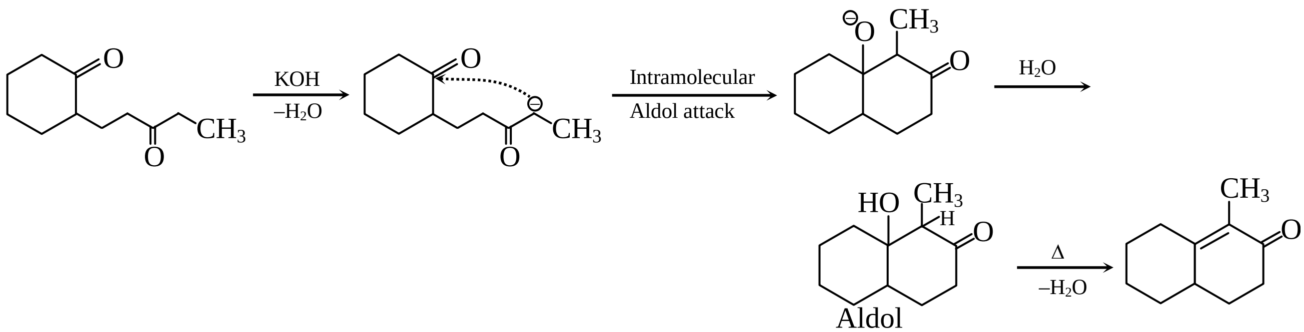
(B) 

(C) 

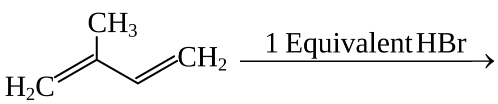
(D) 

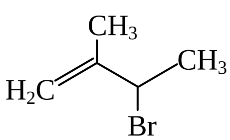
Ans. **[A]**

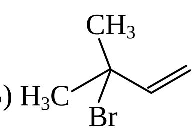
Sol.

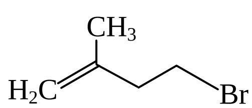


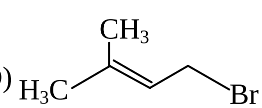
Q.34 In the following reaction, the major product is -



(A) 

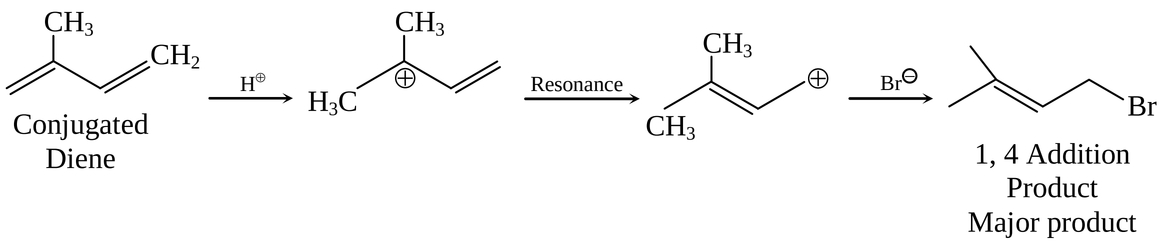
(B) 

(C) 

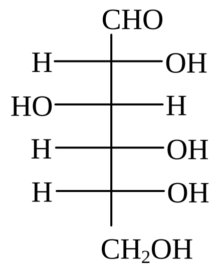
(D) 

Ans. **[D]**

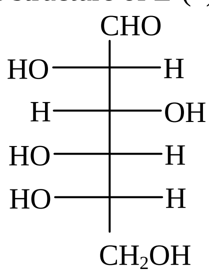
Sol.

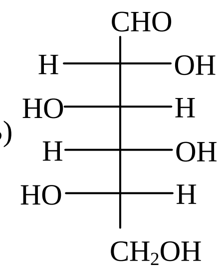


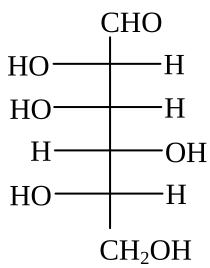
Q.35 The structure of D-(+)-glucose is

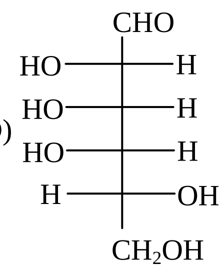


The structure of L-(–)-glucose is :

(A) 

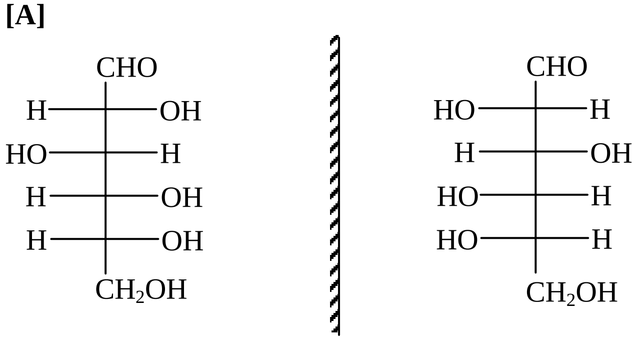
(B) 

(C) 

(D) 

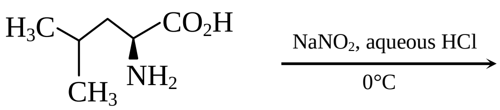
Ans. [A]

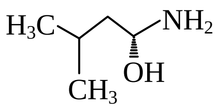
Sol.

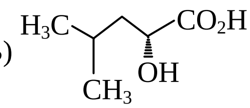


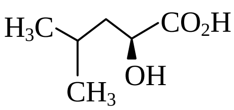
D(+) glucose L(–) glucose

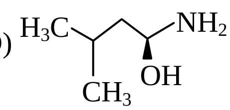
Q.36 The major product of the reaction is



(A) 

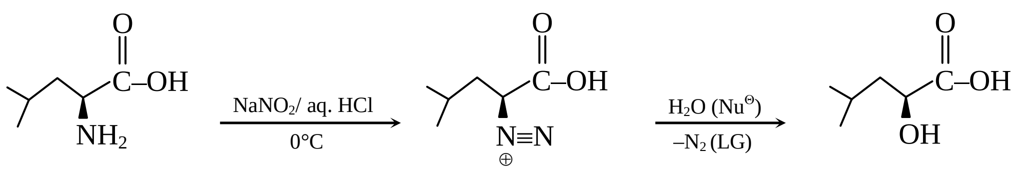
(B) 

(C) 

(D) 

Ans. **[C]**

Sol.



Q.37 The correct statement(s) about Cr^{2+} and Mn^{3+} is (are) -

[Atomic numbers of Cr = 24 and Mn = 25]

(A) Cr^{2+} is a reducing agent

(B) Mn^{3+} is an oxidizing agent

(C) Both Cr^{2+} and Mn^{3+} exhibit d^4 electronic configuration

(D) When Cr^{2+} is used as a reducing agent, the chromium ion attains d^5 electronic configuration

Ans. **[A,B,C]**

Sol. $Cr^{+2} \longrightarrow Cr^{+3}$

d^4 d^3 (stable)

So, act as a reducing agent

$Mn^{+3} \longrightarrow Mn^{+2}$

d^4 d^5 (stable)

So, act as a oxidising agent

Q.38 Copper is purified by electrolytic refining of blister copper. The correct statement(s) about this process is (are) :

(A) Impure Cu strip is used as cathode

(B) Acidified aqueous $CuSO_4$ is used as electrolyte

(C) Pure Cu deposits at cathode

(D) Impurities settle as anode mud

Ans. **[B,C,D]**

Sol. (i) Due to acidified aqueous $CuSO_4$, conduction is increased

(ii) Cathode is made by pure Cu

(iii) Anode is made by Impure Cu

(iv) Ag and Au are settle down as a anode mud

SECTION – 3 (Maximum Marks : 16)

- This section contains **TWO** questions
- Each question contains two columns, **Column I** and **Column II**
- **Column I** has **four** entries (A), (B), (C) and (D)
- **Column II** has **five** entries (P), (Q), (R), (S) and (T)
- Match the entries in **Column I** with the entries in **Column II**
- One or more entries in **Column I** may match with one or more entries in **Column II**
- The ORS contains a 4 × 5 matrix whose layout will be similar to the one shown below:

(A)	(P)	(Q)	(R)	(S)	(T)
(B)	(P)	(Q)	(R)	(S)	(T)
(C)	(P)	(Q)	(R)	(S)	(T)
(D)	(P)	(Q)	(R)	(S)	(T)

- For each entry in **Column I**, darken the bubbles of all the matching entries. For example, if entry (A) in **Column I** matches with entries (Q), (R) and (T), then darken these three bubbles in the ORS. Similarly, for entries (B), (C) and (D).
- **Marking scheme:**

For each entry in **Column I.**

+2 If only the bubble(s) corresponding to all the correct match(es) is(are) darkened

0 If none of the bubbles is darkened

–1 In all other cases

Q.39 Match the anionic species given in **Column - I** that are present in the ore(s) given in **Column - II.**

Column - I	**Column - II**
(A) Carbonate	(P) Siderite
(B) Sulphide	(Q) Malachite
(C) Hydroxide	(R) Bauxite
(D) Oxide	(S) Calamine
	(T) Argentite

Ans. **[A → P,Q,S ; B → T ; C → Q,R ; D → R]**

Sol. Siderite → $FeCO_3$

Malachite → $CuCO_3 . Cu(OH)_2$

Bauxite → $AlO_x (OH)_{3-2x}$ $(0 < x < 1)$

Calamine → $ZnCO_3$

Argentite → Ag_2S

Q.40 Match the thermodynamic processes given under **Column - I** with the expressions given under **Column - II**

Column - I	**Column - II**
(A) Freezing of water at 273 K and 1 atm	(P) $q = 0$
(B) Expansion of 1 mol of an ideal gas into a vacuum under isolated conditions	(Q) $w = 0$
(C) Mixing of equal volumes of two ideal gases at constant temperature and pressure in an isolated container	(R) $\Delta S_{sys} < 0$
(D) Reversible heating of $H_2(g)$ at 1 atm from 300 K to 600 K, followed by reversible cooling to 300 K at 1 atm	(S) $\Delta U = 0$
	(T) $\Delta G = 0$

Ans. **[A → R,T ; B → P,Q,S ; C → P,Q,S ; D → P,Q,S,T]**

Sol. (A) Water $\underset{273K}{\overset{1atm}{\rightleftharpoons}}$ Ice

* From liquid to solid entropy decreases

$\therefore \Delta S < 0$

* Liquid is present under equilibrium with solid

$\therefore \Delta G = 0$

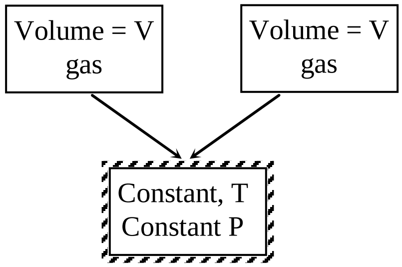
Therefore A → R,T

(B) Expansion of 1 mole gas into vacuum under isolated condition

* Isolated system q = 0
* During expansion against vacuum no work is done W = U
* No change in internal energy $\Delta E = 0$

Therefore $B \rightarrow P,Q,S$

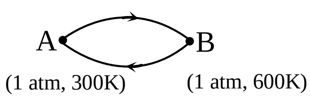
(C)



$\therefore$ q = 0, W = 0, $\Delta U = 0$

$\therefore C \rightarrow P,Q,S$

(D) Reversible cyclic process



$W_{AB} = -nR(600-300) = -300\ nR$

$W_{BA} = -nR(300-600) = 300\ nR$

$\therefore \sum W = 0$

* For cyclic process change in the value of state functions are zero i.e. $\Delta U = 0, \Delta G = 0$
* From 1st law : $\sum W = 0, \Delta U = 0, \therefore \sum q = 0$

$\therefore D \rightarrow P, Q, S, T$

PART III : MATHEMATICS

SECTION 1 (Maximum Marks : 32)

- This Section contains EIGHT questions
- The answer to each questions is a **SINGLE DIGIT INTEGER** ranging from 0 to 9, both inclusive
- For each questions, darken the bubble corresponding to the correct integer in the ORS
- Marking scheme :

  + 4 If the bubble corresponding to the answer is darkened

  0 In all other cases

Q.41 The minimum number of times a fair coin needs to be tossed, so that the probability of getting at least two heads is at least 0.96, is

Ans. **[8]**

Sol. 1 – P(no head) – P(one head) ≥ 0.96

$$1 - {}^{n}C_0\left(\frac{1}{2}\right)^n - {}^{n}C_1\left(\frac{1}{2}\right)^n \ge 0.96$$

$$\frac{n+1}{2^n} \le 0.04$$

$$\frac{n+1}{2^n} \le \frac{1}{25}$$

Minimum value of n is 8.

Q.42 Let *n* be the number of ways in which 5 boys and 5 girls can stand in a queue is such a way that all the girls stand consecutively in the queue. Let *m* be the number of ways in which 5 boys and 5 girls can stand in a queue in such a way that exactly four girls stand consecutively in the queue. Then the value of $\frac{m}{n}$ is

Ans. **[5]**

Sol. n = 5! 6!

$$m = {}^{5}C_4 \times 5! \times {}^{6}C_2 \times 2! \times 4!$$

${}^{6}C_2$ → selecting 2 gaps in between 5 boys

$5!$ → arranging 5 boys

${}^{5}C_4$ → selecting 4 girls

$$m = 5! \times 5! \times {}^{6}C_2 \times 2!$$

$$\therefore \frac{m}{n} = \frac{{}^{6}C_2 \times 2!}{6} = 5$$

Q.43 If the normals of the parabola $y^2 = 4x$ drawn at the end points of its latus rectum are tangents to the circle $(x-3)^2 + (y+2)^2 = r^2$, then the value of r^2 is

Ans. **[2]**

Sol. One end point of the latus rectum is (1, 2)

Slope of the normal $= -\left(\frac{y}{2}\right)_{(1,2)} = -1$

So its equation will be

$y - 2 = -(x - 1)$

$x + y = 3$

it is tangent to the circle

so $\left|\frac{3-2-3}{\sqrt{2}}\right| = r$

$\therefore\ r^2 = 2$

Q.44 Let $f : R \to R$ be a function defined by $f(x) = \begin{cases} [x], & x \le 2 \\ 0, & x > 2 \end{cases}$,

where [x] is the greatest integer less than or equal to x. If $I = \int_{-1}^{2} \frac{xf(x^2)}{2+f(x+1)} dx,$ then the value of $(4I - 1)$ is

Ans. **[0]**

Sol. $I = \int_{-1}^{\sqrt{2}} \frac{x[x^2]}{2+f(x+1)} dx + \int_{\sqrt{2}}^{2} 0\, dx$

$= \int_{-1}^{0} \frac{x(0)}{2+0} dx + \int_{0}^{1} \frac{x(0)}{2+1} dx + \int_{1}^{\sqrt{2}} \frac{x(1)}{2+0} dx$

$= \frac{1}{2}\int_{1}^{\sqrt{2}} x\, dx = \left[\frac{x^2}{4}\right]_{1}^{\sqrt{2}} = \frac{1}{4}$

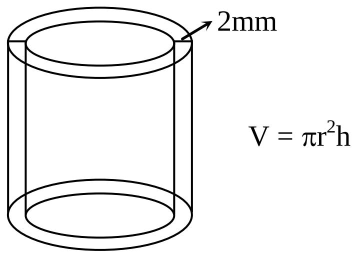
$\therefore\ 4I - 1 = 0$

Q.45 A cylindrical container is to be made from certain solide material with the following constraints : It has a fixed inner volume of V mm^3, has a 2 mm thick solid wall and is open at the top. The bottom of the container is a solid circular disc of thickness 2 mm and is of radius equal to the outer radius of the container.

If the volume of the material used to make the container is minimum when the inner radius of the container is 10 mm, then the value of $\frac{V}{250\pi}$ is

Ans. **[4]**

Sol.



Let S denotes volume of the material used to make the container

$S = \pi r^2 \times 2 + 2\pi rh \times 2$

$S = 2\pi r^2 + 4\pi r \times \frac{V}{\pi r^2}$

$S = 2\pi r^2 + \frac{4V}{r}$

Now $\frac{dS}{dr} = 4\pi r - \frac{4V}{r^2}$

$r^3 = \frac{V}{\pi}$

$r = \left(\frac{V}{\pi}\right)^{1/3}$

Given r = 10 mm

$V = 1000\pi$

$\therefore \frac{V}{250\pi} = 4$

Q.46 Let $F(x) = \int_{x}^{x^2+\frac{\pi}{6}} 2\cos^2 t\, dt$ for all $x \in R$ and $f : \left[0, \frac{1}{2}\right] \to [0, \infty)$ be a continuous function. For $a \in \left[0, \frac{1}{2}\right]$, if F'(a) + 2 is the area of the region bounded by x = 0, y = 0, y = f(x) and x = a, then f(0) is

Ans. **[3]**

Sol. $F'(x) = 2\cos^2\left(x^2 + \frac{\pi}{6}\right).2x - 2\cos^2 x$

Now $\int_0^a f(x)\,dx = 4a\cos^2\left(a^2 + \frac{\pi}{6}\right) - 2\cos^2 a + 2$

Differentiate with respect to 'a'

$\Rightarrow f(a) = 4\cos^2\left(a^2 + \frac{\pi}{6}\right) + 4a\left(2\cos\left(a^2 + \frac{\pi}{6}\right)\right)\left(-\sin\left(a^2 + \frac{\pi}{6}\right)\right).2a + 4\cos a \sin a$

$\therefore f(0) = 4\cos^2\frac{\pi}{6} + 0 + 0$

$= 4 \times \frac{3}{4} = 3$

Q.47 The number of distinct solutions of the equation

$$\frac{5}{4}\cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2$$

in the interval [0, 2π] is

Ans. **[8]**

Sol. $\frac{5}{4}\cos^2 2x + 1 - 2\sin^2 x\cos^2 x + 1 - 3\sin^2 x\cos^2 x = 2$

$\Rightarrow 5\cos^2 2x - 5\sin^2 2x = 0$

$\Rightarrow 5\cos 4x = 0$

$\Rightarrow 4x = 2n\pi \pm \frac{\pi}{2}, n \in I$

$\Rightarrow x = \frac{n\pi}{2} \pm \frac{\pi}{8}, n \in I$

In the interval [0, 2π]

Possible solutions are

$\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$

So total number of solutions in [0, 2π] is 8.

Q.48 Let the curve C be the mirror image of the parabola $y^2 = 4x$ with respect of the line $x + y + 4 = 0$. If A and B are the points of intersection of C with the line $y = -5$, then the distance between A and B is

Ans. [4]

Sol. Let any point on the parabola $y^2 = 4x$ is $(t^2, 2t)$

Let its mirror image with respect to the line $x + y + 4 = 0$ is (h, k) then

$$\frac{h-t^2}{1} = \frac{k-2t}{1} = \frac{-2(t^2+2t+4)}{2}$$

$\therefore h = -2t - 4,\ k = -t^2 - 4$

So $k + 4 = -\left(\frac{h+4}{2}\right)^2$

$\Rightarrow (h + 4)^2 = -4(k + 4)$

So locus of C is

$(x + 4)^2 = -4(y + 4)$

It intersects $y = -5$

So $(x + 4)^2 = 4$

$\Rightarrow x + 4 = \pm 2$

$\Rightarrow x = -2, -6$

$\therefore |x_1 - x_2| = 4$

SECTION 2 (Maximum Marks : 40)

- This Section contains **TEN** questions
- Each question has **FOUR** options (A), (B), (C) and (D) **ONE OR MORE THAN ONE** of these four option(s) is (are) correct
- For each questions, darken the bubble(s) corresponding to all the correct option (s) in the **ORS**
- Marking scheme :

+4 If only the bubble (s) corresponding to all the correct option (s) is (are) darkened

0 If none of the bubbles is darkened

–2 In all other cases

Q.49 Let g : R → R be a differentiable function with g(0) = 0, g'(0) = 0 and g'(1) ≠ 0. Let

$$f(x) = \begin{cases} \frac{x}{|x|}g(x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

and $h(x) = e^{|x|}$ for all x ∈ R. Let (f o h) (x) denote f(h(x)) and (h o f)(x) denote h(f(x)). Then which of the following is (are) true ?

(A) f is differentiable at x = 0

(B) h is differentiable at x = 0

(C) f o h is differentiable at x = 0

(D) h o f is differentiable at x = 0

Ans. **[A,D]**

Sol. g(0) = 0, $g'(0) = 0$, $g'(1) \neq 0$

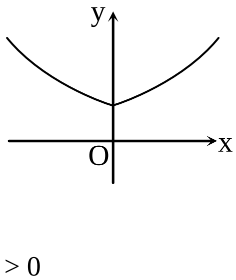
$$f(x) = \begin{cases} g(x) & x > 0 \\ 0 & x = 0 \\ -g(x) & x < 0 \end{cases}$$

$$f'(x) = \begin{cases} g'(x) & x > 0 \\ 0 & x = 0 \\ -g'(x) & x < 0 \end{cases}$$

f(x) is diff. at x = 0, as $g'(0) = 0$

So option (A) is true.

$h(x) = e^{|x|}$



not diff. at x = 0

So option (B) is not true

f o h(x) = f(h(x)) as h(x) > 0

So f(h(x)) = g(h(x))

$\frac{df(h(x))}{dx} = g'(h(x))\, h'(x)$ since $g'(1) \neq 0$

at x = 0, f(h(x)) is not differentiable

So option (C) is not true

h o f (x) = h(f(x))

$= e^{|f(x)|}$

Differentiable at x = 0

So option (D) is true.

Q.50 Let $f(x) = \sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)$ for all $x \in R$ and $g(x) = \frac{\pi}{2}\sin x$ for all $x \in R$. Let (f o g)(x) denote f(g(x)) and (g o f)(x) denote g(f(x)). Then which of the following is (are) true ?

(A) Range of f is $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(B) Range of f o g is $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(C) $\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{\pi}{6}$

(D) There is an $x \in R$ such that (g o f)(x) = 1

Ans. **[A,B,C]**

Sol. $f(x) = \sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)$

$-1 \leq \sin x \leq 1$ as $x \in R$

$-\frac{\pi}{2} \leq \frac{\pi}{2}\sin x \leq \frac{\pi}{2}$

$-1 \leq \sin\left(\frac{\pi}{2}\sin x\right) \leq 1$

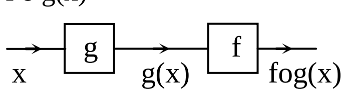
$-\frac{\pi}{6} \leq \frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right) \leq \frac{\pi}{6}$

$-\frac{1}{2} \leq \sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right) \leq \frac{1}{2}$

Range $\in \left[-\frac{1}{2}, \frac{1}{2}\right]$

So option (A) is correct.

f o g(x)



Range of g(x) = $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ ∩ Domain of f(x) = R

$$\text{Common} = \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

Range of fog(x) = Range of f(x) when input $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$$= \left[-\frac{1}{2},\frac{1}{2}\right]$$

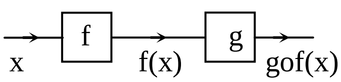
So option (B) is correct

$$\lim_{x\to 0}\frac{\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)}{\frac{\pi}{2}\sin x} = \lim_{x\to 0}\left(\frac{\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)}{\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)}\right)\left(\frac{\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)}{\left(\frac{\pi}{2}\sin x\right)}\right)$$

$$= \lim_{x\to 0}\frac{\pi}{6}.(1) = \frac{\pi}{6}$$

So option (B) is correct

g o f(x)



Range of f(x) = $\left[-\frac{1}{2},\frac{1}{2}\right]$

$$\text{g o f(x)} = \frac{\pi}{2}\sin f(x)$$

$$= \frac{\pi}{2}\sin\underbrace{\left(\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)\right)}_{-\frac{1}{2}\text{ to }\frac{1}{2}} = 1$$

$$\sin\left(-\frac{1}{2}\text{to}\frac{1}{2}\right) = \frac{2}{\pi} \approx 0.6379 \text{ (not possible)}$$

$$\frac{1}{2} \approx 28.5^{o}$$

So option (D) is not correct.

Q.51 Let ΔPQR be a triangle Let $\vec{a} = \overrightarrow{QR}$, $\vec{b} = \overrightarrow{RP}$ and $\vec{c} = \overrightarrow{PQ}$. If $|\vec{a}| = 12$, $|\vec{b}| = 4\sqrt{3}$ and $\vec{b}.\vec{c} = 24$, then which of the following is (are) true ?

(A) $\frac{|\vec{c}|^2}{2} - |\vec{a}| = 12$

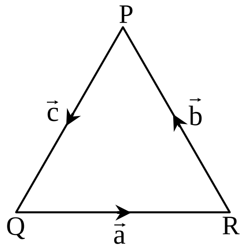
(B) $\frac{|\vec{c}|^2}{2} + |\vec{a}| = 30$

(C) $|\vec{a}\times\vec{b} + \vec{c}\times\vec{a}| = 48\sqrt{3}$

(D) $\vec{a}\cdot\vec{b} = -72$

Ans. **[A, C, D]**

Sol.



$\vec{b}\cdot\vec{c} = 24$

$\vec{a} + \vec{b} + \vec{c} = 0$

$\vec{a} = -\vec{b} - \vec{c}$

$144 = 48 + |\vec{c}|^2 + 2\vec{b}\cdot\vec{c}$

$96 = |\vec{c}|^2 + 2(24)$

$|\vec{c}|^2 = 48$

(A) $\frac{|\vec{c}|^2}{2} - |\vec{a}| = \frac{48}{2} - 12 = 12$

So option (A) is correct

(B) $\frac{|\vec{c}|^2}{2} + |\vec{a}| = \frac{48}{2} + 12 = 24$

So option (B) is not correct

(C) $|\vec{a}\times\vec{b} + \vec{c}\times\vec{a}| = |\vec{a}\times\vec{b} - \vec{a}\times\vec{c}|$

$= |\vec{a}\times(\vec{b} - \vec{c})|$

$= |(-\vec{b} - \vec{c})\times(\vec{b} - \vec{c})|$

$= |0 + \vec{b}\times\vec{c} - \vec{c}\times\vec{b} + 0|$

$= 2|\vec{b}\times\vec{c}|$

$= 2\left(\sqrt{|b|^2|c|^2 - (\vec{b}\cdot\vec{c})^2}\right)$

$= 2\sqrt{48(48) - (24)^2}$

$= 2 \times 24 \times \sqrt{3} = 48\sqrt{3}$

So option (C) is correct

(D) $\vec{a}+\vec{b}+\vec{c}=0$

$\vec{a}+\vec{b}=-\vec{c}$

$144+48+2\vec{a}\cdot\vec{b}=48$

$\vec{a}\cdot\vec{b}=-72$

So option (D) is correct

Q.52 Let X and Y be two arbitrary, 3 × 3, non-zero, skew-symmetric matrices and Z be an arbitrary 3 × 3, non-zero, symmetric matrix. Then which of the following matrices is (are) skew symmetric ?

(A) $Y^3Z^4 - Z^4Y^3$

(B) $X^{44} + Y^{44}$

(C) $X^4Z^3 - Z^3X^4$

(D) $X^{23} + Y^{23}$

Ans. **[C,D]**

Sol. $X^T = -X Y^T = -Y Z^T = Z$

(A) $(Y^3 Z^4 - Z^4 Y^3)^T = (Y^3 Z^4)^T - (Z^4 Y^3)^T$

$= (Z^4)^T (Y^3)^T - (Y^3)^T (Z^4)^T$

$= (Z^T)^4 (Y^T)^3 - (Y^T)^3 (Z^T)^4$

$= Z^4 (-Y)^3 - (-Y)^3 (Z)^4$

$= -Z^4 Y^3 + Y^3 Z^4$

$= Y^3Z^4 - Z^4Y^3$

Hence it is symmetric matrix.

(B) $(X^{44} + Y^{44})^T = (X^T)^{44} + (Y^T)^{44}$

$= X^{44} + Y^{44}$

Hence it is symmetric matrix.

(C) $(X^4 Z^3 - Z^3 X^4)^T = (X^4 Z^3)^T - (Z^3 X^4)^T$

$= (Z^T)^3 (X^T)^4 - (X^T)^4 (Z^T)^3$

$= Z^3 X^4 - X^4 Z^3$

$= -(X^4 Z^3 - Z^3 X^4)$

Hence it is skew symmetric matrix.

(D) $(X^{23} + Y^{23})^T = (X^T)^{23} + (Y^T)^{23}$

$= -(X^{23} + Y^{23})$

Hence it is skew symmetric matrix.

Q.53 Which of the following values of α satisfy the equation.

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\alpha ?$$

(A) – 4 (B) 9

(C) – 9 (D) 4

Ans. **[B,C]**

Sol.
$$\begin{vmatrix} 1 & 2 & 1 \\ 4 & 4 & 1 \\ 9 & 6 & 1 \end{vmatrix}\begin{vmatrix} 1 & 1 & 1 \\ \alpha & 2\alpha & 3\alpha \\ \alpha^2 & 4\alpha^2 & 9\alpha^2 \end{vmatrix} = -648\alpha$$

$\Rightarrow$ $(-4)(2\alpha^3) = -648\alpha$

$\Rightarrow$ $\alpha^2 = 81$

$\Rightarrow$ $\alpha = \pm 9$

Q.54 In R^3, consider the planes $P_1 : y = 0$ and $P_2 : x + z = 1$. Let P_3 be a plane, different from P_1 and P_2, which passes through the intersection of P_1 and P_2. If the distance of the point (0, 1, 0) from P_3 is 1 and the distance of a point (α, β, γ) from P_3 is 2, then which of the following relations is (are) true ?

(A) $2\alpha + \beta + 2\gamma + 2 = 0$ (B) $2\alpha - \beta + 2\gamma + 4 = 0$

(C) $2\alpha + \beta - 2\gamma - 10 = 0$ (D) $2\alpha - \beta + 2\gamma - 8 = 0$

Ans. **[B,D]**

Sol. Equation of P_3 will be

$(x + z - 1) + \lambda y = 0$

$x + \lambda y + z - 1 = 0$

Its disantace from (0, 1, 0) will be

$$\left|\frac{0+\lambda+0-1}{\sqrt{1+\lambda^2+1}}\right| = 1$$

$\Rightarrow (\lambda - 1)^2 = 1 + \lambda^2 + 1$

$\Rightarrow \lambda^2 - 2\lambda + 1 = \lambda^2 + 2$

$\Rightarrow \lambda = -1/2$

$\therefore$ Equation of P_3 is $2x - y + 2z - 2 = 0$

Its distance from (α, β, γ) is

$$\frac{|2\alpha - \beta + 2\gamma - 2|}{3} = 2$$

$2\alpha - \beta + 2\gamma - 2 = \pm 6$

$\Rightarrow 2\alpha - \beta + 2\gamma - 8 = 0$

and $2\alpha - \beta + 2\gamma + 4 = 0$

Q.55 In R^3, Let be a straight line passing through the origin. Suppose that all the points on L are at a constant distance from the two planes $P_1 : x + 2y - z + 1 = 0$ and $P_2 : 2x - y + z - 1 = 0$. Let M be the locus of the feet of the perpendiculars drawn from the points on L to the plane P_1. Which of the following points lie(s) on M ?

(A) $\left(0, -\frac{5}{6}, -\frac{2}{3}\right)$ (B) $\left(-\frac{1}{6}, -\frac{1}{3}, \frac{1}{6}\right)$

(C) $\left(-\frac{5}{6}, 0, \frac{1}{6}\right)$ (D) $\left(-\frac{1}{3}, 0, \frac{2}{3}\right)$

Ans. **[A,B]**

Sol. $P_1 : x + 2y - z + 1 = 0$

& $P_2 : 2x - y + z - 1 = 0$

Direction Ratios of common line $(1, -3, -5) \Rightarrow \hat{i} - 3\hat{j} - 5\hat{k}$

$$L : \frac{x}{1} = \frac{y}{-3} = \frac{z}{-5} = t$$

Let $M(\alpha, \beta, \gamma)$ is feet of perpendicular from $(t, -3t, -5t)$ on P_1

$$\frac{\alpha - t}{1} = \frac{\beta + 3t}{2} = \frac{\gamma + 5t}{-1} = -\left(\frac{t - 6t + 5t + 1}{6}\right)$$

$$\alpha = t - \frac{1}{6} \qquad \beta = -3t - \frac{1}{3} \qquad \gamma = -5t + \frac{1}{6}$$

Only option (A) & (B) satisfies.

Q.56 Let P and Q be distinct points on the parabola $y^2 = 2x$ such that a circle with PQ as diameter passes through the vertex O of the parabola. If P lies in the first quadrant and the area of the triangle ΔOPQ is $3\sqrt{2}$, then which of the following is (are) the coordinates of P ?

(A) $(4, 2\sqrt{2})$ (B) $(9, 3\sqrt{2})$

(C) $\left(\frac{1}{4}, \frac{1}{\sqrt{2}}\right)$ (D) $(1, \sqrt{2})$

Ans. **[A,D]**

Sol. Let coordinates of P and Q are $P(2t_1^2, 2t_1)$, $Q(2t_2^2, 2t_2)$

As the circle with PQ as diameter passes through the vertex O.

So $\angle POQ = 90^\circ$

$m_{OP} \times m_{OQ} = -1$

$$\frac{2t_1}{2t_1^2} \times \frac{2t_2}{2t_2^2} = -1 \Rightarrow t_1 t_2 = -1$$

Now Area of ΔOPQ will be

$$\frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2t_1^2 & 2t_1 \\ 1 & 2t_2^2 & 2t_2 \end{vmatrix} = 3\sqrt{2}$$

$\Rightarrow 4|t_1 t_2 (t_1 - t_2)| = 6\sqrt{2}$

$\Rightarrow |t_1 - t_2| = \frac{3}{\sqrt{2}}$

$\Rightarrow t_1 + \frac{1}{t_1} = \frac{3}{\sqrt{2}}$

$\Rightarrow \sqrt{2}\ t_1^2 - 3t_1 + \sqrt{2} = 0$

$\Rightarrow (\sqrt{2}\, t_1 - 1)(t_1 - \sqrt{2}) = 0$

$\Rightarrow t_1 = \frac{1}{\sqrt{2}}, \sqrt{2}$

$\therefore$ Coordinates of P can be $(4, 2\sqrt{2})$, $(1, \sqrt{2})$

Q.57 Let y(x) be a solution of the differential equation $(1 + e^x)y' + ye^x = 1$. If $y(0) = 2$, then which of the following statements is (are) true ?

(A) $y(-4) = 0$

(B) $y(-2) = 0$

(C) y(x) has a critical point in the interval (–1, 0)

(D) y(x) has no critical point in the interval (–1, 0)

Ans. **[A,C]**

Sol. $(1 + e^x)y' + ye^x = 1$

$\Rightarrow d((1 + e^x)y) = 1$

$\Rightarrow y(1 + e^x) = x + C$

$\because y(0) = 2$

$\therefore 2 \times 2 = C \Rightarrow C = 4$

$\therefore y(1 + e^x) = x + 4$

$\therefore y(-4) = 0,\ y(-2) = \frac{2}{1+e^{-2}} \neq 0$

For critical point $y' = 0$

$\Rightarrow ye^x = 1$

Now let $g(x) = ye^x - 1 = \frac{e^x(x+4)}{1+e^x} - 1$

$g(-1) = \frac{3e^{-1}}{1+e^{-1}} - 1 = \frac{3}{e+1} - 1 < 0$

$g(0) = 2 - 1 > 0$

So there exists one value of x in (–1, 0) for which $g(x) = 0 \Rightarrow y' = 0$

$\Rightarrow$ there exist a critical point of y(x) in (–1, 0)

Q.58 Consider the family of all circles whose centers lie on the straight line y = x. If this family of circles is represented by the differential equation $Py'' + Qy' + 1 = 0$, where P, Q are functions of x, y and y' (here $y' = \frac{dy}{dx}$, $y'' = \frac{d^2y}{dx^2}$), then which of the following statements is (are) true ?

(A) $P = y + x$

(B) $P = y - x$

(C) $P + Q = 1 - x + y + y' + (y')^2$

(D) $P - Q = x + y - y' - (y')^2$

Ans. **[B, C]**

Sol. Let the equation of circle is

$$(x - \alpha)^2 + (y - \alpha)^2 = r^2$$

$\Rightarrow x^2 + y^2 - 2\alpha x - 2\alpha y + 2\alpha^2 - r^2 = 0$

differentiate w.r.t. x

$\Rightarrow 2x + 2yy' - 2\alpha - 2\alpha y' = 0$

$\Rightarrow \alpha = \frac{x + yy'}{1 + y'}$ ...(i)

differentiate again w.r.t. x

$2 + 2(y')^2 + 2yy'' - 2\alpha y'' = 0$

$\Rightarrow \alpha = \frac{1 + (y')^2 + yy''}{y''}$ ...(ii)

from (i) & (ii)

$xy'' + yy'y'' = 1 + (y')^2 + yy'' + y' + (y')^3 + yy'y''$

$\Rightarrow (y - x)y'' + y'[y' + 1 + (y')^2] + 1 = 0$

$P = y - x$

$Q = y' + 1 + (y')^2$

SECTION 3 (Maximum Marks: 16)

- This section contains **TWO** questions
- Each question contains two columns, **Column I** and **Column II**
- **Column I** has **four** entries (A), (B), (C) and (D)
- **Column II** has **five** entries (P), (Q), (R), (S) and (T)
- Match the entries in **Column I** with the entries in **Column II**
- One or more entries in **Column I** may match with one or more entries in **Column II**
- The ORS contains a 4 × 5 matrix whose layout will be similar to the one shown below:

(A)	(P)	(Q)	(R)	(S)	(T)
(B)	(P)	(Q)	(R)	(S)	(T)
(C)	(P)	(Q)	(R)	(S)	(T)
(D)	(P)	(Q)	(R)	(S)	(T)

- For each entry in **Column I**, darken the bubbles of all the matching entries. For example, if entry (A) in **Column I** matches with entries (Q), (R) and (T), then darken these three bubbles in the ORS. Similarly, for entries (B), (C) and (D).
- **Marking scheme:**

For each entry in **Column I.**

+2 If only the bubble(s) corresponding to all the correct match(es) is(are) darkened

0 If none of the bubbles is darkened

–1 In all other cases

Q.59

	Column I	**Column II**
(A)	In R^2, if the magnitude of the projection vector of the vector $\alpha\hat{i}+\beta\hat{j}$ on $\sqrt{3}\hat{i}+\hat{j}$ is $\sqrt{3}$ and if $\alpha = 2+\sqrt{3}\beta$, then possible value(s) of $\lvert\alpha\rvert$ is (are)	(P) 1
(B)	Let a and b be real numbers such that the function $f(x) = \begin{cases} -3ax^2-2, & x<1 \\ bx+a^2, & x\ge 1 \end{cases}$ is differentiable for all $x \in R$. Then possible value(s) of a is (are)	(Q) 2
(C)	Let $\omega \ne 1$ be a complex cube root of unity. If $(3-3\omega+2\omega^2)^{4n+3}+(2+3\omega-3\omega^2)^{4n+3}+(-3+2\omega+3\omega^2)^{4n+3}=0$ then possible value(s) of n is (are)	(R) 3
(D)	Let the harmonic mean of two positive real numbers a and b be 4. If q is a positive real number such that a, 5, q, b is an arithmetic progression, then the value(s) of $\lvert q-a\rvert$ is (are)	(S) 4
		(T) 5

Ans. [(A) → (P,Q); (B) → (P,Q); (C) → (P,Q,S,T) D → (Q,T)]

Sol. (A) Projection of $\alpha\hat{i}+\beta\hat{j}$ on $\sqrt{3}\,\hat{i}+\hat{j}$ will be

$$\frac{(\alpha\hat{i}+\beta\hat{j}).(\sqrt{3}\,\hat{i}+\hat{j})}{2} = \sqrt{3}$$

$$\Rightarrow \left|\frac{\sqrt{3}\alpha+\beta}{2}\right| = \sqrt{3}$$

as $\alpha = 2 + \sqrt{3}\,\beta$

So $\left|\sqrt{3}\alpha + \frac{\alpha-2}{\sqrt{3}}\right| = 2\sqrt{3}$

$\Rightarrow |3\alpha + \alpha - 2| = 6$

$\Rightarrow 4\alpha - 2 = \pm 6$

$\alpha = 2, -1$

$\therefore |\alpha| = 1, 2$

So (A) $\rightarrow$ (P, Q)

(B) $f(x) = \begin{cases} -3ax^2 - 2, & x < 1 \\ bx + a^2, & x \geq 1 \end{cases}$

as f(x) is differentiable for $x \in R$. So it will also be continuous for $x \in R$.

So $-3a - 2 = b + a^2$

$a^2 + 3a + 2 + b = 0$ … (i)

$f'(x) = \begin{cases} -6ax, & x < 1 \\ b, & x \geq 1 \end{cases}$

So $-6a = b$ … (ii)

From (i) & (ii)

$a^2 + 3a - 6a + 2 = 0$

$\Rightarrow a = 1, 2$

(B) $\rightarrow$ (P,Q)

(C) $(3 - 3\omega + 2\omega^2)^{4n+3} + (2 + 3\omega - 3\omega^2)^{4n+3} + (-3 + 2\omega + 3\omega^2)^{4n+3} = 0$

$\Rightarrow (1 - 5\omega)^{4n+3} + (\omega - 5\omega^2)^{4n+3} + (-5 + \omega^2)^{4n+3} = 0$

$\Rightarrow (1 - 5\omega)^{4n+3} + \omega^{4n+3}(1 - 5\omega)^{4n+3} + (\omega^2)^{4n+3}(1 - 5\omega)^{4n+3} = 0$

$\Rightarrow (1 - 5\omega)^{4n+3} + (1 + \omega^{4n+3} + (\omega^2)^{4n+3}) = 0$

$\Rightarrow 1 + \omega^{4n+3} + (\omega^2)^{4n+3} = 0$

$\Rightarrow$ 4n + 3 should not be integral multiple of 3

So n can be 1, 2, 4, 5

So (C) $\rightarrow$ (P,Q,S,T)

(D) $\frac{2ab}{a+b} = 4$

also a, 5, q, b are in A.P.

then a = 5 – d, b = 5 + 2d,

So $\frac{2(5-d)(5+2d)}{5-d+5+2d} = 4$

$\Rightarrow 40 + 4d = 2(25 + 5d - 2d^2)$

$\Rightarrow 2d^2 - 3d - 5 = 0$

$\Rightarrow d = -1, 5/2$

$|q - a| = |2d| = 2, 5$

(D) → (Q,T)

Q.60 **Column I** **Column II**

(A) In a triangle ΔXYZ, let a, b and c be the lengths of the sides opposite to the angles X, Y and Z, respectively. If $2(a^2 - b^2) = c^2$ and $\lambda = \frac{\sin(X-Y)}{\sin Z}$, then possible values of n for which $\cos(n\pi\lambda) = 0$ is (are) — (P) 1

(B) In a triangle ΔXYZ, let a, b and c be the lengths of the sides opposite to the angles X, Y and Z, repsectively. If 1 + cos 2X – 2 cos 2Y = 2 sin X sin Y, then possible value(s) of $\frac{a}{b}$ is (are) — (Q) 2

(C) In R^2, let $\sqrt{3}\hat{i} + \hat{j}$, $\hat{i} + \sqrt{3}\hat{j}$ and $\beta\hat{i} + (1-\beta)\hat{j}$ be the position vectors of X, Y and Z with respect to the origin O, respectively. If the distance of Z from the bisector of the acute angle of $\overrightarrow{OX}$ with $\overrightarrow{OY}$ is $\frac{3}{\sqrt{2}}$, then possible value(s) of |β| is (are) — (R) 3

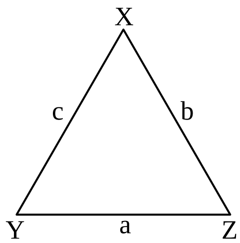
(D) Suppose that $F(\alpha)$ denotes the area of the region bounded by $x = 0$, $x = 2$, $y^2 = 4x$ and $y = |\alpha x - 1| + |\alpha x - 2| + \alpha x$, where $\alpha \in \{0, 1\}$. Then the value(s) of $F(\alpha) + \frac{8}{3}\sqrt{2}$, when $\alpha = 0$ and $\alpha = 1$, is (are)

(S) 5

(T) 6

Ans. **[A → (P, R, S); B →(P); C →(P, Q); D →(S, T)]**

Sol. **(A)**



$2(a^2 - b^2) = c^2$

Using Sine Rule

$\Rightarrow 2(\sin^2 X - \sin^2 Y) = \sin^2 Z$

$\Rightarrow 2\sin (X - Y) \sin (X + Y) = \sin^2 Z$

$\Rightarrow 2\sin (X - Y) \sin Z - \sin^2 Z = 0$

$\Rightarrow \sin Z (2 \sin (X - Y) - \sin Z) = 0$

$\Rightarrow \sin Z = 0 \quad \text{or} \quad \sin(X - Y) = \frac{1}{2} \sin z$

$\therefore \quad \lambda = \frac{\sin(X - Y)}{\sin Z} = \frac{1}{2}$

Now $\cos\left(\frac{n\pi}{2}\right) = 0$

For $n = 1, 3, 5$

$\therefore$ (A) → (P, R, S)

(B) $1 + \cos 2X - 2 \cos 2Y = 2 \sin X \sin Y$

$\Rightarrow 2 \cos^2 X - 2(2 \cos^2 Y - 1) = 2 \sin x \sin Y$

$\Rightarrow 2 \cos^2 X + 2 - 4 \cos^2 Y = 2 \sin X \sin Y$

$\Rightarrow 2 - 2 \sin^2 X + 2 - 4 + 4 \sin^2 Y = 2 \sin X \sin Y$

$\Rightarrow 2 \sin^2 X + 2 \sin X \sin Y - 4 \sin^2 Y = 0$

Using Sine Rule

$a^2 + ab - 2b^2 = 0$

$$\Rightarrow \left(\frac{a}{b}\right)^2 + \frac{a}{b} - 2 = 0 \qquad \Rightarrow \frac{a}{b} = 1, -2$$

$\therefore$ (B) $\longrightarrow$ (P)

(C) Bisector of OX and OY will be along the vector $(\sqrt{3}+1)\hat{i} + (1+\sqrt{3})\hat{j}$

$\therefore \quad x = y$

$\Rightarrow \quad x - y = 0$

$$\left|\frac{\beta - (1-\beta)}{\sqrt{2}}\right| = \frac{3}{\sqrt{2}}$$

$\therefore$ $|2\beta - 1| = 3$

$2\beta - 1 = \pm 3$

$\therefore$ $\beta = 2, -1$

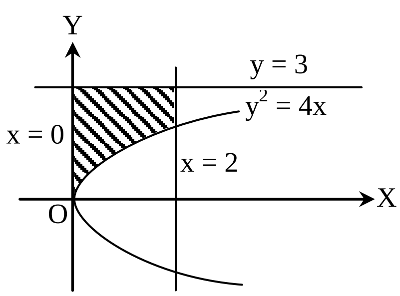
$\therefore$ $|\beta| = 1, 2$

(C) $\rightarrow$ (P, Q)

(D) When $\alpha = 0$

$y = |0 - 1| + |0 - 2| + 0 = 3$

then Area bounded by $x = 0$, $x = 2$, $y^2 = 4x$, and $y = 3$ will be



$$F(0) = 6 - \int_0^2 \sqrt{4x}\, dx$$

$$= 6 - 2\left[\frac{x^{3/2}}{3/2}\right]_0^2 = 6 - \frac{8\sqrt{2}}{3}$$

$$\therefore \quad F(0) + \frac{8\sqrt{2}}{3} = 6$$

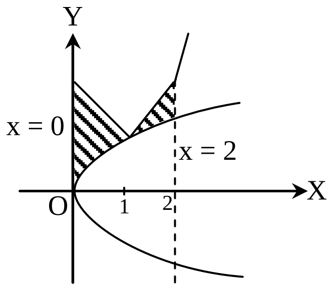
when $\alpha = 1$

$y = |x - 1| + |x - 2| + x$

$$= \begin{cases} 3 - x \ ; \ 0 \le x < 1 \\ x + 1 \ ; \ 1 \le x < 2 \\ 3x - 3 ; \quad x \ge 2 \end{cases}$$

then Area bounded by x = 0, x = 2, $y^2 = 4x$ and

y = | x – 1 | + | x – 2 | + x   will be



$$F(1) = 5 - \int_0^2 \sqrt{4x}\, dx$$

$$= 5 - \frac{8\sqrt{2}}{3}$$

$$\therefore \ F(1) + \frac{8\sqrt{2}}{3} = 5$$

(D) → (S, T)